String theory lecture - Exercise sheet 5

To be discussed on November 19th

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The goal of this exercise sheet is to explore further the residual symmetries of the Polyakov action after imposing flat gauge (the conformal diffeomorphisms) to motivate the lightcone gauge quantization. After that we derive the Lorentz generators and compute their commutation relations. Eventually we compute the anomaly arising in these relations when working in lightcone gauge.

1 Conformal diffeomorphisms

a) A conformal Killing vector ϵ^a satisfies the conformal Killing equation

$$\nabla_a \epsilon_b + \nabla_b \epsilon_a - h_{ab} \nabla^c \epsilon_c = 0\,. \tag{1.1}$$

Explain what is the effect of a diffeomorphism of the form

$$\xi^a \to \xi'^a = \xi^a + \epsilon^a(\xi)\,. \tag{1.2}$$

What is the relevance of these conformal diffeomorphisms in string theory[1]?

b) Show that a conformal Killing vector leads to a conserved current $J^a \equiv T^{ab}\epsilon_b$

c) Using lightcone worldsheet coordinates, show that the conformal Killing vectors generate the transformations

$$\xi^\pm \to \xi'^\pm = \xi^\pm + \epsilon^\pm(\xi^\pm)\,. \tag{1.3}$$

Note: From this we deduce that the conformal diffeomorphisms correspond to $\text{Diff}(S^1)^2$ (right-moving and left-moving) and we expect the Witt algebra (or Virasoro algebra in the quantum theory) to arise. Let us explicitly show this in what follows.

d) First compute the following Poisson brackets (denoted $[\cdot,\cdot]_{\text{PB}}$):

$$[T_{\pm\pm}(\tau,\sigma), X^\mu(\tau,\sigma')]_{\text{PB}}\,. \tag{1.4}$$

To do this use that $T_{\pm\pm} = -\frac{1}{\alpha'}\partial_\pm X \cdot \partial_\pm X$ and the equal-time brackets:

$$\begin{aligned} [X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')]_{\text{PB}} &= [\dot{X}^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')]_{\text{PB}} = 0\,, \\ [X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')]_{\text{PB}} &= \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma')\,. \end{aligned} \tag{1.5}$$

[1] Note that the specific form of the term proportional to the metric in (1.1) is such that the expression is traceless.

e) From the conserved current we deduce the following conserved charges for the closed string:

$$L_{\epsilon^\pm} = -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma \epsilon^\pm(\xi^\pm) T_{\pm\pm}(\xi^\pm)\,. \tag{1.6}$$

Compute $[L_{\epsilon^\pm}, X^\mu(\tau,\sigma)]_{\mathrm{PB}}$ and conclude that these charges indeed generate the conformal transformations.

f) Decompose the Killing vector field into Fourier modes

$$\epsilon^\pm(\xi^\pm) = \sum_{m\in\mathbb{Z}} \epsilon^\pm_m e^{2i\pi m\xi^\pm/l}\,, \tag{1.7}$$

and give an expression for the modes $L^\pm_m$ defined such that

$$L_{\epsilon^\pm} \equiv \sum_{m\in\mathbb{Z}} \epsilon^\pm_m L^\pm_m\,. \tag{1.8}$$

g) From the result of the previous question, you see that these $L^\pm_m$ are the modes of the energy-momentum tensor. Compute the brackets $[T_{\pm\pm}(\tau,\sigma), T_{\pm\pm}(\tau,\sigma')]_{\mathrm{PB}}$ and $[L^\pm_m, L^\pm_n]_{\mathrm{PB}}$ to uncover the two expected Witt algebras.

Note: The bottom line of this exercise is that we showed that the residual gauge freedom in our Polyakov action is generated by the Fourier modes of the energy-momentum tensor. This leads to the idea of lightcone gauge: Imposing the constraint $T_{ab} = 0$ should be equivalent to further gauge fix the residual freedom before quantizing. We then expect that, in lightcone gauge quantization, there will be no constraint to impose and the space of physical states should be easier to obtain. This is true up to the level-matching condition for the closed string, which arises in lightcone gauge from the identification of the zero modes α^μ_0 and $\tilde{\alpha}^\mu_0$.

2 Lorentz generators

One downside of lightcone gauge quantization is the lack of explicit Lorentz invariance. Anomalies can arise and asking them to vanish constrains the number of dimensions D and the parameter a. This is what we investigate in this exercise by looking at the Lorentz generators and their algebra.

a) Compute the conserved charges associated with Poincaré invariance of the classical string. To do this follow these steps:

- Write the Lorentz transformations at the infinitesimal level and show that the two-index generator is antisymmetric. You can refer to the book by Zwiebach, A First Course in String Theory, section 8.5.
- Apply Noether theorem to derive the conserved worldsheet current from which you can express the conserved charges $J^{\mu\nu}$ given by

$$J^{\mu\nu} = T\int_0^l \mathrm{d}\sigma (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)\,. \tag{2.1}$$

  You can refer to section 8.2 of the book by Zwiebach for a reminder of Noether theorem in field theories.

b) For the open string with NN boundary conditions, show that one can write

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu}\,, \tag{2.2}$$

with

$$l^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu\,, \qquad E^{\mu\nu} \equiv -i\sum_{n=1}^{+\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n)\,, \tag{2.3}$$

where $E^{\mu\nu}$ is written in normal ordered form. Argue why we can safely write $E^{\mu\nu}$ like this thanks to $J^{\mu\nu}$ being free from ordering ambiguities.

c) Show that the Lorentz algebra is indeed satisfied:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho}J^{\nu\sigma} + i\eta^{\nu\sigma}J^{\mu\rho} - i\eta^{\mu\sigma}J^{\nu\rho} - i\eta^{\nu\rho}J^{\mu\sigma}\,. \tag{2.4}$$

d) In lightcone gauge, the gauge fixing choice is not Lorentz invariant and this implies that a Lorentz transformation that preserves the gauge condition acts non-linearly on the coordinates (cf Green–Schwarz–Witten, p97). A consequence of this is that the Lorentz algebra in lightcone gauge can suffer from anomalies and we need to cancel them for the theory to make sense. One can argue that the anomaly term takes the form

$$[J^{i-}, J^{j-}] = -\frac{1}{\alpha'(p^+)^2}\sum_{m=1}^{+\infty}(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m)\Delta_m\,, \tag{2.5}$$

where Δ_m is to be determined and asked to vanish. To do that, let us follow Green–Schwarz–Witten, p99 (be careful though cause I think there are typos in the book):

- Show that $[x^-, 1/p^+] = i/(p^+)^2$ and $[x^i, E^j] = -iE^{ij}$, where $E^j \equiv p^+E^{j-}$ (set $2\alpha' = 1$ for convenience).
- From these commutators, show that you can write

  $$[J^{i-}, J^{j-}] = -(p^+)^{-2}C^{ij} \quad \text{with} \quad C^{ij} = 2ip^+\alpha^-_0 E^{ij} - [E^i, E^j] + ip^iE^j - ip^jE^i\,. \tag{2.6}$$

  Tips:
  - You will need $[x^i, p^-] = ip^i/p^+$ and $[x^-, p^-] = ip^-/p^+$.
  - Introduce an identity $(p^+)^{-1}p^+$ before each E^{i-} and E^{j-} to make E^i and E^j appear. This is convenient because E^i and E^j then do not depend on p^+.
- From the form of the anomaly in (2.5), show that

  $$\langle 0|\,\alpha^k_m C^{ij}\alpha^l_{-m}\,|0\rangle = m^2(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il})\Delta_m\,. \tag{2.7}$$

- Now we want to compute explicitly $\langle 0|\,\alpha^k_m C^{ij}\alpha^l_{-m}\,|0\rangle$ in order to identify Δ_m. First compute the two following useful commutators for $m > 0$:

  $$\begin{aligned}[] [\alpha^k_m, E^{ij}] &= -i\delta^{ik}\alpha^j_m - (i \leftrightarrow j)\,,\\ [\alpha^k_m, E^{j-}] &= -i\delta^{kj}\alpha^-_m - i\sum_{n>0}\frac{m}{np^+}\alpha^j_{-n}\alpha^k_{m+n} + i\sum_{n>0}\frac{m}{np^+}\alpha^k_{m-n}\alpha^j_n\,.\end{aligned} \tag{2.8}$$

  Remember that we have

  $$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m+n}\,, \qquad [\alpha^i_m, \alpha^-_n] = m\alpha^i_{m+n}/p^+\,. \tag{2.9}$$

e) With these commutators you can express $\langle 0| \alpha_m^k C^{ij} \alpha_{-m}^l |0\rangle$ and identify

$$\Delta_m = m\frac{26-D}{12} + \frac{2}{m}\left(\frac{D-2}{24} + a\right) , \tag{2.10}$$

where a is defined like in the lecture (for the open string and with $2\alpha' = 1$):

$$p^+ p^- = \frac{1}{2}p_\perp^2 + N_\perp + a \, . \tag{2.11}$$

To arrive at the result, take your time to compute each contribution before gathering everything. You will at some point face a commutator $[p^+\alpha_m^-, p^+\alpha_{-m}^-]$ that you can evaluate using the Virasoro algebra with its anomaly that we computed in a previous exercise sheet. Note however that the operator $p^+\alpha_m^-$ only contains transverse oscillators, so you should replace $D \to D-2$ in the expression of the anomaly. Also, the ordering constant a introduced in the definition of $p^+\alpha_m^-$ shifts the coefficient c_1 in $A(m) = c_1 m + c_3 m^3$ that we evaluated by $-2a$. The relevant anomaly is thus given by

$$A(m) = \frac{D-2}{12}m(m^2-1) - 2am \, . \tag{2.12}$$