String theory lecture - Exercise sheet 3

To be discussed on November 5^{th}

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The goal of this exercise sheet is to uncover the mode expansions for various types of strings and boundary conditions. We also reproduce carefully the derivation of the constraints on the first excited open-string state. Eventually we derive the anomaly term in the Virasoro algebra.

1 Neumann/Dirichlet boundary conditions

As you have seen during the lecture, the equation of motion for the string in light-cone coordinates is given by

$$\partial_+\partial_- X^\mu(\sigma^+,\sigma^-) = 0\,, \tag{1.1}$$

which is solved by

$$X^\mu(\sigma^+,\sigma^-) = X^\mu_{\mathrm{L}}(\sigma^+) + X^\mu_{\mathrm{R}}(\sigma^-)\,. \tag{1.2}$$

a) Derive the mode expansion $X^\mu_{\mathrm{NN}}(\tau,\sigma)$ for an open sting with Neumann–Neumann boundary conditions. To do this, go through the following steps:

- Imposing $\partial_\sigma X^\mu_{\mathrm{NN}}(\tau,\sigma)|_{\sigma=0} = 0$, relate X_{L} and X_{R}.
- Use the other boundary condition at $\sigma = l$ to deduce that $\partial_\pm X^\mu_{\mathrm{L}}$ is periodic with period $2l$.
- Deduce that you can Fourier expand with the following conventions:

$$\partial_\pm X^\mu_{\mathrm{L}}(\xi^\pm) = \frac{\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} \alpha^\mu_n e^{-\frac{\pi i}{l}n\sigma^\pm}\,, \tag{1.3}$$

and integrate to find the mode expansion of $X^\mu_{\mathrm{NN}}(\tau,\sigma)$.

b) Follow the same strategy to find $X^\mu_{\mathrm{DD}}(\tau,\sigma)$ by imposing Dirichlet boundary conditions at both $\sigma = 0$ and $\sigma = l$.

c) Now find $X^\mu_{\mathrm{DN}}(\tau,\sigma)$ by imposing a Dirichlet condition at $\sigma = 0$ and a Neumann condition at $\sigma = l$.

Note: From all these expansions, one can compute the position of the center of mass and the total momentum of the string for the different boundary conditions. If you do this, you find in particular that the NN string is moving with constant velocity while the DD one is not.

2 Tilt

The various boundary conditions studied in the previous exercise can be chosen independently for the different X^μ. For the directions with Dirichlet conditions, we interpret this as having the endpoints of the string being confined on a brane extended along these directions. (Note that full Neumann conditions can be interpreted as the endpoints being attached to spacetime-filling branes.) Now consider the boundary term

$$\int \mathrm{d}\tau \partial_\sigma X^\mu \delta X_\mu \Big|_{\sigma=0}^{\sigma=l} = 0\,, \tag{2.1}$$

and restrict to only two spatial directions, $\mu = 1, 2$, for simplicity. Clearly, this is satisfied for, e.g. Neumann conditions for X^1 and Dirichlet for X^2. Now, using the obvious $SO(2)$ symmetry of this setting before boundary conditions are imposed, discuss a possible generalization corresponding to a "tilted" brane.

3 Antiperidocity

Describe the mode expansion of the antiperiodic boson satisfying

$$X^\mu(\tau, \sigma + l) = -X^\mu(\tau, \sigma)\,. \tag{3.1}$$

Note: This is the relevant boundary condition for the twisted sector of a non-freely acting orbifold.

4 First excited level

As you have seen during the lectures, the first excited level for the open string is written as

$$\xi_\mu \alpha_{-1}^\mu \left|0, p\right\rangle\,, \tag{4.1}$$

with ξ_μ a polarization vector. Redo carefully the derivation of the L_1 constraint on this state.

5 The Virasoro anomaly

The L_m's satisfy the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n}\,, \tag{5.1}$$

where the second term is the anomaly. To compute the anomaly, we follow the steps outlined in the book by Green, Schwarz and Witten (section 2.2.2).

a) Use the Jacobi identity

$$[L_m, [L_n, L_k]] + \text{cyclic perm.} = 0\,, \tag{5.2}$$

with $m + n + k = 0$ and then set $k = 1$ to show that

$$A(n+1) = \frac{n+2}{n-1}A(n) - \frac{2n+1}{n-1}A(1)\,. \tag{5.3}$$

b) Assuming that $A(n)$ is a polynomial, use the recursion relation to show that $A(n) = c_1 n + c_3 n^3$ with $c_1, c_3 \in \mathbb{R}$.

c) Fix c_1 and c_3 by computing the value of $\langle 0| [L_m, L_{-m}] |0\rangle$ for two smartly chosen values of m.