

String theory lecture - Exercise sheet 2

To be discussed on October 29th

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The goal of this exercise sheet is to manipulate the Polyakov action for strings and start playing with the quantized theory.

1 Polyakov action

The Polyakov action for the bosonic string is

$$S_p = -\frac{T}{2} \int_{\Sigma} d^2\xi \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \quad (1.1)$$

a) Check that the action is invariant under

- diffeomorphisms $\xi^a \rightarrow \xi'^a(\xi^0, \xi^1)$,
- Poincaré transformations $X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu + V^\mu$ with $\Lambda \in SO(1, D-1)$,
- Weyl rescalings $h_{ab}(\xi) \rightarrow h'_{ab} = \phi(\xi) h_{ab}(\xi)$.

b) For a matrix A such that $\det(A) \neq 0$, show that we have the relation

$$\delta(\det A) = (\det A) \text{Tr}(A^{-1} \delta A), \quad (1.2)$$

by following two methods:

- Using the identity $\ln \det A = \text{Tr} \ln A$,
 - Using the explicit formula for $\det A$ in terms of the Levi-Civita symbol.
- c) Go carefully through the derivation of the stress-energy tensor, as presented in the lecture, to find

$$T_{ab} \equiv \frac{4\pi}{\sqrt{-h}} \frac{\delta S_p}{\delta h^{ab}} = -2\pi T \left(G_{ab} - \frac{1}{2} h_{ab} G_{cd} h^{cd} \right), \quad (1.3)$$

where $G_{ab} \equiv \partial_a X^\mu \partial_b X_\mu$.

d) The equations of motion then tell you that $T^{ab} = 0$. Show that the trace $T_a{}^a$ vanishes even before imposing the equations of motion. Derive again $T_a{}^a = 0$ without even using the explicit form of T_{ab} , but using instead one of the symmetries of the action.

2 Flat gauge

- Given the symmetries of the Riemann tensor, show that of the 2^4 components of R_{abcd} in two dimensions, only one is independent.
- Check that the ansatz $R_{abcd} = \lambda(h_{ac}h_{bd} - h_{ad}h_{bc})$ is consistent with the symmetries of the Riemann tensor. Show that $\lambda = \mathcal{R}/2$ with \mathcal{R} the Ricci scalar.
- Compute the Einstein tensor and interpret the result.

Note: As you saw in the lecture, one can show that under a Weyl rescaling, $h_{ab} \rightarrow h'_{ab} = e^{2\omega(\xi^0, \xi^1)} h_{ab}$ we have the following transformation:

$$\sqrt{-h'} \mathcal{R}' \rightarrow \sqrt{-h} (\mathcal{R} - 2\nabla^2 \omega). \quad (2.1)$$

This is what allows one to go to a gauge where the metric has a zero Ricci scalar everywhere. From the structure of the Riemann tensor in two dimensions, this makes it possible to go to the flat gauge where $h_{ab} = \eta_{ab}$.

3 Mode expansion

The mode expansion for the closed string of length l that you have seen in the lecture is given by

$$X^\mu = x^\mu + \frac{\pi\alpha'}{l} p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-2i\pi n \sigma^+ / l} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-2i\pi n \sigma^- / l}, \quad (3.1)$$

with $\sigma^\pm \equiv \tau \pm \sigma$.

- Derive the commutation relations of x^μ , p^μ , α_n^μ and $\tilde{\alpha}_n^\mu$ displayed in the lectures from the canonical commutation relations of X^μ and Π^μ . To do this, use the relation

$$\int_0^l d\sigma \int_0^l d\sigma' e^{2\pi i m \sigma / l} e^{2\pi i n \sigma' / l} \delta(\sigma - \sigma') = l \delta_{m+n}. \quad (3.2)$$

- Derive the classical expression for the Hamiltonian

$$H = \frac{\pi}{l} \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n + \frac{\pi}{l} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n, \quad (3.3)$$

from the Polyakov action (do not worry about ordering issues). Here $\alpha_{-n} \cdot \alpha_n \equiv \alpha_{-n}^\mu \alpha_n^\nu \eta_{\mu\nu}$.