String theory lecture - Exercise sheet 13

To be discussed on January 28th

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The goal of this exercise sheet is to first manipulate the RNS superstring action in light-cone coordinates to check its SUSY invariance and make explicit the possibility to have periodic or anti-periodic boundary conditions for the fermions. In the second exercise, we study the circle compactification of the bosonic string to discover new stringy effects, anticipate the existence of *T-duality* and exhibit the phenomenon of *gauge enhancement*.

1 RNS superstring action

The superstring action in flat gauge takes the form

$$S = -\frac{1}{4\pi}\int \mathrm{d}^2\xi \left(\frac{1}{\alpha'}\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \gamma^a \partial_a \psi_\mu\right) . \tag{1.1}$$

a) Using the spinor component notation $\psi \equiv (\psi_-, \psi_+)^T$, show that in light-cone coordinates the action takes the form you have seen in the lecture (use the same conventions defined there for the Clifford algebra and the gamma matrices):

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\xi \left(\frac{2}{\alpha'}\partial_+ X^\mu \partial_- X_\mu + i(\psi_-^\mu \partial_+ \psi_{-\mu} + \psi_+^\mu \partial_- \psi_{+\mu})\right) . \tag{1.2}$$

b) The on-shell SUSY transformations are given by

$$\sqrt{\frac{2}{\alpha'}}\delta X^\mu = \bar{\epsilon}\psi^\mu\,, \qquad \delta\psi^\mu = -\sqrt{\frac{2}{\alpha'}}\frac{i}{2}\gamma^a \partial_a X^\mu \epsilon\,, \tag{1.3}$$

with the Majorana spinor $\epsilon \equiv (\epsilon_-, \epsilon_+)^T$.

Fix $\alpha' = 2$ for convenience and show that in light-cone coordinates these transformations become

$$\delta X^\mu = i(\epsilon_+\psi_- - \epsilon_-\psi_+)\,, \qquad \delta\psi_\pm^\mu = \pm\epsilon_\mp \partial_\pm X^\mu\,, \tag{1.4}$$

c) It is convenient to define $\eta^- \equiv \epsilon_+$ and $\eta^+ = -\epsilon_-$. Rewrite the transformations with this parameter. Consider then the η^+ transformation (chose $\eta^- = 0$) and prove the invariance of the action.

d) Consider the closed string, vary the action of $\psi_\pm^\mu$ and integrate by part. Read off the equation of motion and show explicitly that the variation of the action vanishes both for periodic and anti-periodic boundary conditions (you may assume that $\delta\psi_\pm^\mu|_{\tau=\pm\infty} = 0$).

e) The right-moving fermion ψ_+^μ has the following mode decomposition:

$$\psi_+^\mu(\tau,\sigma) = \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} \tilde{\psi}_r^\mu e^{-2i\pi r\sigma^+/l} , \quad (\phi = 0 \text{ for R}, \ \phi = \frac{1}{2} \text{ for NS}). \tag{1.5}$$

The conjugate variable is $\Pi_+^\mu = \frac{i}{2\pi}\psi_+^\mu$. Using the canonical equal-time relation

$$\{\psi_+^\mu(\tau,\sigma), \Pi_+^\mu(\tau,\sigma')\} = i\delta(\sigma-\sigma')\eta^{\mu\nu} , \tag{1.6}$$

show that the Fourier modes satisfy

$$\{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \delta_{r+s}\eta^{\mu\nu} . \tag{1.7}$$

2 Circle compactification

Consider the theory of a massless scalar field $\phi(x^M)$ in $d+1$ spacetime dimensions. We compactify the $(d+1)^{\text{th}}$ dimension on a circle of radius R. It means we identify $x^d \cong x^d + 2\pi R$ and our spacetime geometry is $\mathbb{R}^{1,d-1} \times S^1(R)$. Splitting the spacetime index like $M \equiv \{\mu, d\}$ with $\mu = 0, \ldots, d-1$, we can expand the field along the compact direction like

$$\phi(x^M) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) \exp\left(\frac{inx^d}{R}\right) . \tag{2.1}$$

a) What are the eigenvalues of the momentum operator in the compact direction?

b) Starting from the equation of motion in $d+1$ dimensions, show that from the effective d-dimensional point of view the modes $\phi_n(x^\mu)$ form an infinite tower of fields with mass-squared $m^2 = -p_\mu p^\mu = \frac{n^2}{R^2}$. These modes are called the *Kaluza–Klein (KK) modes* and n is the *KK number*.

c) Now we turn to the bosonic string with a target space featuring one compact dimension (that we take to be the 25^{th} direction), such that $X^{25} \cong X^{25} + 2\pi R$. Argue that the periodicity condition for the closed string (of length 2π) can then be generalized like

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi R w , \qquad w \in \mathbb{Z} . \tag{2.2}$$

The new integer w is called *winding number*. Give a geometric interpretation of this number (you can draw a picture).

d) Recall what is the most general solution of $\partial_+\partial_- X^{25}(\tau,\sigma) = 0$ for the closed string by introducing arbitrary left and right zero modes α_0^{25} and $\tilde{\alpha}_0^{25}$. Do not impose any periodicity yet.

e) Impose the periodicity condition (2.2) to find a relation between α_0^{25} and $\tilde{\alpha}_0^{25}$.

f) Use your knowledge about the KK modes to constrain the center or mass momentum given by $(\alpha_0^{25} + \tilde{\alpha}_0^{25})/\sqrt{2\alpha'}$. Now you can fully express α_0^{25} and $\tilde{\alpha}_0^{25}$ as well as the mode expansion in terms of the KK and winding numbers.

Let us recap what we obtained so far: The 25^{th} coordinate is expanded like

$$X^{25}(\tau,\sigma) = x^{25} + \sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}_0^{25} + \alpha_0^{25})\tau + \sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}_0^{25} - \alpha_0^{25})\sigma + N + \tilde{N}\,, \tag{2.3}$$

with

$$\tilde{\alpha}_0^{25} + \alpha_0^{25} = \sqrt{2\alpha'}\frac{n}{R}\,, \qquad \tilde{\alpha}_0^{25} - \alpha_0^{25} = \frac{2Rw}{\sqrt{2\alpha'}}\,. \tag{2.4}$$

(For literature treating the circle compactification, you can refer to the book by Becker, Becker, Schwarz, section 6.1 or Timo Weigand's lecture notes section 7.2.)

g) From the Virasoro constraints $(L_0 - 1)\,|\text{phys}\rangle = 0$ and $(\tilde{L}_0 - 1)\,|\text{phys}\rangle = 0$, derive the following level-matching condition and effective mass-squared in 25 dimensions

$$\begin{aligned} N - \tilde{N} &= nw\,, \\ \alpha' m^2 &= \alpha'\left[\frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2}\right] + 2N + 2\tilde{N} - 4\,. \end{aligned} \tag{2.5}$$

h) What happens under the transformation

$$n \leftrightarrow w\,, \qquad R \to R' = \frac{\alpha'}{R}\,? \tag{2.6}$$

Try to gain physical understanding of the situation and derive the value of the radius which is a fixed point of the transformation. It is called *self-dual radius*.

i) Let us now look at the spectrum. The ground state with $n = w = 0$ and no oscillator gives the usual tachyon with $m^2 = -4/\alpha'$. At the massless level with $N = \tilde{N} = 1$ and $n = w = 0$, write the possible states by acting either with α_{-1}^{μ}, $\mu = 0, \ldots, 24$ or with α_{-1}^{25} (and their tilde counterparts). You should find the 25-dimensional graviton as well as two 25-dimensional vectors and one scalar. Interpret this in terms of a gauge group.

j) Now we consider the sector with $n = w = \pm 1$. The level-matching condition is modified to $N = \tilde{N} + 1$. Write the spectrum obtained with $\tilde{N} = 0$ and $N = 1$. You should find one vector and one scalar for $n = w = 1$ as well as one vector and one scalar for $n = w = -1$. Calculate the mass of these states.

k) You would find another copy of these states by looking at the sector with $n = -w = \pm 1$. What happens to their mass at the self-dual radius defined above? Interpret again (recall what happens in the open-string case at the special point where the positions of two D-branes coincide).