String theory lecture - Exercise sheet 11

To be discussed on January 14$^{\text{th}}$

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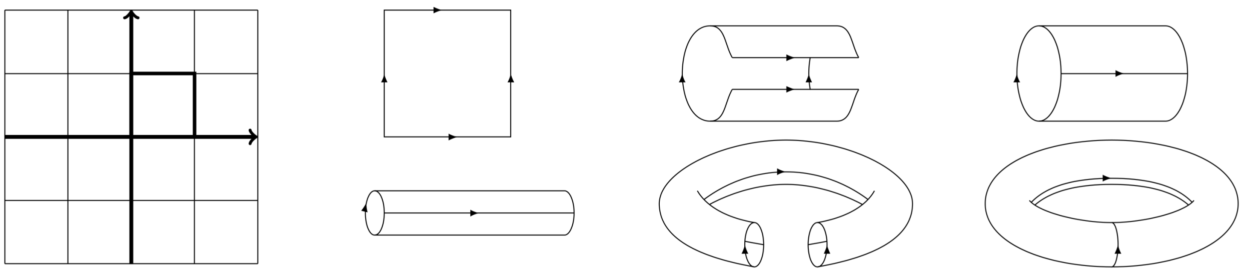
The goal of this exercise sheet is to compute the closed-string one-loop vacuum amplitude. We prepare our dive into the computation by exploring the moduli space of the torus. After that, we derive the partition function of the closed string and study its essential properties by comparing it with a field theory result. For the derivation of the partition function you can refer to the lecture notes by Timo Weigand and David Tong that are available on the lecture website.

1 One-loop warm-up: The torus moduli space

Before we evaluate amplitudes at one-loop for the closed string, we need to know how to integrate over conformally inequivalent metrics on the torus: We need to uncover the *torus moduli space*. The torus has by definition the topology of $S^1 \times S^1$ and is parametrized by

$$(\sigma_1, \sigma_2) \cong (\sigma_1, \sigma_2) + (m, n)\,, \qquad m, n \in \mathbb{Z}\,. \tag{1.1}$$

This identification gives rise to a lattice in $\mathbb{R}^2$ that describes the torus (see image below).



a) The *Riemann-Roch theorem* relates the number of metric moduli μ of a Riemann surface to the number of conformal Killing vectors κ. The theorem states

$$\mu - \kappa = -3\chi\,. \tag{1.2}$$

- Argue that the theorem is in agreement with what you know about the sphere S^2.
- The torus being simply the product of two circles, guess what its conformal group is and deduce from the Riemann-Roch theorem that it should have only one complex modulus.

b) We already know (see lecture and exercise sheet 2) that conformal transformations in 2d allow us locally to make our space flat. A non-trivial topology generally prevents the map to flat space from being well-defined globally but, because $\chi = 0$ for the torus, in this case it is.

- Defining $z \equiv \sigma_1 + i\sigma_2$ one can thus realize the metric $\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}$. But in the process, the original periodicity may be modified to shifts by arbitrary vectors u_a and v_a, $a \in \{1,2\}$. Let the lattice of the torus defined above be generated by two such vectors u_a and v_a instead of $(1,0)$ and $(0,1)$. Argue that you can fix u_a without loss of generality to be $(1,0)$. Then you are left only with v_a.
- Define $\tau \equiv v_1 + iv_2$: This is the complex modulus we were looking for. The torus has the flat metric $\mathrm{d}z\mathrm{d}\bar{z}$ but the periodicity involves the modulus. Another way to look at the situation is to rewrite the coordinate like $z = \sigma_1 + \tau\sigma_2$ to recover the original periodicities. Perform this rewriting explicitly,

$$\mathrm{d}s^2 = \underbrace{\mathrm{d}z\mathrm{d}\bar{z}}_{\tau \text{ in periodicties}} = \underbrace{|\mathrm{d}\sigma_1 + \tau\mathrm{d}\sigma_2|^2}_{\text{orginial periodicities}} , \tag{1.3}$$

  and give the elements of the τ-dependent metric.

c) We have identified the torus complex modulus but it does not mean that different values of τ correspond to inequivalent tori. We must get rid of redundancies to uncover the torus *moduli space*.

- Show that the transformations

$$S: \ \tau \to \tau + 1 , \quad T: \ \tau \to -\frac{1}{\tau} , \quad U: \ \tau \to \frac{\tau}{\tau+1} , \tag{1.4}$$

  appropriately combined with conformal transformations on z, leave the lattice invariant. As a result, values of τ related by S, T and U describe the same torus up to conformal transformations.
- With these transformations (actually with only S and T) you can generate the action

$$\tau \to \frac{a\tau + b}{c\tau + d} , \quad \text{where} \quad ad - bc = 1 , \quad a,b,c,d \in \mathbb{Z} . \tag{1.5}$$

  We have already encountered similar sets of transformations. What group does it correspond to (remember to mod out by the transformation $(a,b,c,d) \to (-a,-b,-c,-d)$)?

d) Now we know which τ-values really describe inequivalent tori. They form the moduli space of the torus. Use the S and T transformations to show that any τ in the upper-half plane can be brought to the *fundamental domain* $\mathcal{F}$ (the lower-half plane is obviously redundant):

$$\mathcal{F} \equiv \left\{ \tau \in \mathbb{C} , \ -\frac{1}{2} \leq \mathrm{Re}\, \tau \leq \frac{1}{2} , \ |\tau| \geq 1 \right\} . \tag{1.6}$$

Represent $\mathcal{F}$ graphically with proper identifications of its boundaries.

2 The partition function for closed strings

a) In this exercise we want to compute the simplest one-loop amplitude Z_{T^2}, i.e. without any operator insertion. The amplitude involves an integration over the torus moduli space and we first need to define a proper integration measure. One needs the notion of distance between metrics on a manifold M which in general takes the form

$$\mathrm{d}s^2_{\text{metric}} \equiv \int_M \sqrt{\det h}\,\delta h_{ab}\delta h_{cd} h^{bc} h^{da}\,. \tag{2.1}$$

Specialized to the torus case we have

$$\mathrm{d}s^2_{\text{metric}} = \int_{T^2} \sqrt{\det h}\,\mathrm{Tr}\left(\partial_\tau h h^{-1} \partial_{\bar\tau} h h^{-1}\right) \mathrm{d}\tau \mathrm{d}\bar\tau\,. \tag{2.2}$$

With the metric on the torus uncovered in the previous exercise, compute this to show that (ignoring prefactors)

$$\mathrm{d}s^2_{\text{metric}} \sim \frac{\mathrm{d}\tau\mathrm{d}\bar\tau}{\tau_2}\,. \tag{2.3}$$

b) It turns out that in the one-loop computation this mathematically natural way of integrating over metrics appear. As a result:

$$Z_{T^2} \sim \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} \langle 1 \rangle_{\tau, \otimes_{i=1}^D X^i} \cdot F(\tau) \equiv \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} Z(\tau) \cdot F(\tau)\,. \tag{2.4}$$

We will focus on the X-CFT piece $Z(\tau) \equiv \langle 1 \rangle_{\tau, \otimes_{i=1}^D X^i}$ in what follows. The factor $F(\tau)$ is the result of two contributions: One from the ghost path integral and the other from a proper division by the conformal killing group volume of the torus. We proceed with the computation of $Z(\tau)$ and we will simply give the result for $F(\tau)$ later.

Assume first a rectangular torus, i.e. $\tau = i\tau_2$, and interpret the path integral $\int \mathcal{D}X e^{-S_X}$ as follows: Start with any state, evolve it by Euclidean time $2\pi\tau_2$ with the Hamiltonian H, take the scalar product with the same state. Sum over states. Translate this into a formula and interpret it as a partition function.

c) Now consider the generic case $\tau = \tau_1 + i\tau_2$. As seen in the previous exercise, this corresponds to a tilt in the torus lattice. Thus, interpret the presence of τ_1 as a spatial translation implemented by a translation operator P before multiplying with the initial state and modify the formula for $Z(\tau)$ accordingly.

d) The Hamiltonian on the cylinder and the translation operator are given by

$$H = \frac{2\pi}{l}(L_0 + \bar L_0 - \frac{c + \bar c}{24})\,, \qquad P = \frac{2\pi}{l}(L_0 - \bar L_0)\,. \tag{2.5}$$

Fix $l = 1$ and focus on only one direction with $c = \bar c = 1$ to find

$$Z(\tau)^{(1)} = \mathrm{Tr}\, q^{L_0^{(1)} - 1/24} \bar q^{\bar L_0^{(1)} - 1/24}\,, \qquad \text{with} \qquad q \equiv e^{2i\pi\tau}\,. \tag{2.6}$$

The superscript (1) simply denotes that we only consider one dimension.

e) Using

$$L_0^{(1)} = \frac{\alpha'}{4}p^2 + N^{(1)}\,, \qquad \bar{L}_0^{(1)} = \frac{\alpha'}{4}p^2 + \bar{N}^{(1)}\,, \tag{2.7}$$

Split the trace into a momentum integral and a trace denoted Tr' which corresponds to summing over oscillators only to obtain

$$Z(\tau)^{(1)} = V_1 (q\bar{q})^{-\frac{1}{24}} \int \frac{\mathrm{d}p}{2\pi} e^{-\pi\tau_2\alpha' p^2} \mathrm{Tr}'\, q^{N^{(1)}} \bar{q}^{\bar{N}^{(1)}}\,, \tag{2.8}$$

where $V_1 \equiv \delta(p-p)$ is the spacetime volume for one direction.

f) Evaluate the trace and compute the Gaussian integral to get

$$Z(\tau)^{(1)} = V_1 \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \frac{1}{(q\bar{q})^{1/24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n}\,. \tag{2.9}$$

g) Combine the contributions from the 26 dimensions and use the fact that the contributions from the ghost part together with the conformal killing group volume of the torus yield the factor $F(\tau) = |\eta(\tau)|^4$ to obtain

$$Z_{\tau_2} \sim iV_{26} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} (4\pi^2\alpha'\tau_2)^{-13} |\eta(\tau)|^{-48}\,, \tag{2.10}$$

where the Dedekind η function is defined as

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)\,. \tag{2.11}$$

At an intuitive level, what is the net effect of the modification of the power of the Dedekind function introduced by the ghosts?

Note for later that if we keep the trace and take the ghosts into account we can write

$$Z_{T^2} = iV_D \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{D}{2}} \frac{1}{q\bar{q}} \mathrm{Tr}'\, q^{N_\perp} \bar{q}^{\bar{N}_\perp}\,, \tag{2.12}$$

where $N_\perp$ counts the transverse oscillators.

h) A way to convince oneself that the expression given for $F(\tau)$ is actually the only possibility that makes sense is to check the *modular invariance* of the partition function, i.e. invariance under $PSL(2,\mathbb{Z})$. By this we simply mean that the integral in (2.10) does not change if we integrate over any domain $\mathcal{F}'$ which is a $PSL(2,\mathbb{Z})$ image of the fundamental domain $\mathcal{F}$. To show this property, manipulate (2.10) to have explicitly a measure like

$$\frac{\mathrm{d}^2\tau}{\tau_2^2}\,, \tag{2.13}$$

and show that it is by itself modular invariant. Then, show that what remains in (2.10) is also modular invariant by making use the following modular properties of the Dedekind function:

$$\eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}} \eta(\tau)\,, \qquad \eta(\tau+1) = e^{i\frac{\pi}{12}} \eta(\tau)\,. \tag{2.14}$$

i) To emphasize some key properties of the partition function, let us compare it with its field theory counterpart. Consider a free massive scalar field ϕ with mass m in D dimensions. Recall that the ϕ path integral gives

$$Z \equiv \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int \mathrm{d}^D x \phi(-\partial^2 + m^2)\phi\right) = \exp\left(-\frac{V_D}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D} \ln\left[\frac{1}{2}(p^2+m^2)\right]\right) . \quad (2.15)$$

j) From this we extract the vacuum amplitude Z_{S^1} for a single particle, analogue to our torus partition function:

$$Z_{S^1} = -\frac{V_D}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D} \ln\left[\frac{1}{2}(p^2+m^2)\right] . \quad (2.16)$$

Using the Schwinger parametrization

$$\frac{1}{x} = \int \mathrm{d}l e^{-lx} \quad \Longrightarrow \quad -\ln(x) = \int_0^{+\infty} \mathrm{d}l \frac{e^{-xl}}{l} + \text{ const.} , \quad (2.17)$$

we get

$$Z_{S^1} = V_D \int \frac{\mathrm{d}^D p}{(2\pi)^D} \int_0^{+\infty} \frac{\mathrm{d}l}{2l} e^{-\frac{1}{2}(p^2+m^2)l} . \quad (2.18)$$

Note that we only care about the m^2-dependence. Doing this manipulation, the UV divergence at high p has been mapped to a divergence at small l. Interpret the different elements in this formula in a way analogous to our string computation. In particular, what are the roles of $\frac{1}{2}(p^2+m^2)$, l in the exponential and $\int_0^{+\infty}\frac{\mathrm{d}l}{2l}$?

k) Perform the p integral and take into account an infinite spectrum like we have in string theory by introducing a trace to get

$$Z_{S^1} = iV_D \int_0^{+\infty} \frac{\mathrm{d}l}{2l^{1+\frac{D}{2}}} (2\pi)^{-\frac{D}{2}} \mathrm{Tr}' \, e^{-m^2 l/2} . \quad (2.19)$$

l) For the strings we have

$$m^2 = \frac{2}{\alpha'}(N_\perp + \bar{N}_\perp - 2) , \qquad N_\perp = \bar{N}_\perp . \quad (2.20)$$

To match the partition function with the string expression, implement the level-matching by introducing

$$\delta_{L_0,\bar{L}_0} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}s e^{2i\pi s(N_\perp - \bar{N}_\perp)} . \quad (2.21)$$

and define $\tau \equiv s + \frac{2li}{\alpha'}$. You should get

$$Z_{S^1} = iV_D \int_{\text{strip}} \frac{\mathrm{d}^2\tau}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{D}{2}} \frac{1}{q\bar{q}} \mathrm{Tr}' \, q^{N_\perp} \bar{q}^{\bar{N}_\perp} . \quad (2.22)$$

This is exactly what we had for the string, with the crucial difference that the integration is over the strip $\{\tau \in \mathbb{C} , \ \tau_2 \geq 0 , \ |\tau_1| \leq \frac{1}{2}\}$.

m) Draw the strip and the fundamental domain on a same graphic. What happens to the integrand when $\tau_2 \to 0$? Conclude about the uttermost importance of modular invariance in string theory.