String theory lecture - Exercise sheet 10

To be discussed on January 7^{th}

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The goal of this exercise sheet is to first study the group of globally well-defined conformal transformations on the sphere and in particular the property allowing to map any distinct three points to any other distinct three points. We then explore in detail the Gaussian integral involved in the tree-level scattering of n closed-string tachyons. Eventually, we manipulate the Virasoro-Shapiro amplitude to prove that it can be nicely rewritten as you have seen in the lecture with Euler functions.

1 Conformal transformations on the sphere

The special conformal transformations that you have seen in the lecture can have zeroes in the denominator. This is why it is important to distinguish between the group of inifinitesimal conformal transformations that you are used to and the group of globally defined conformal diffeomorphisms. This global group depends on the topology of the space under consideration and we focus here on the 2d sphere $S^2 \cong \mathbb{C} \cup \infty$.

a) The generators of the infinitesimal conformal transformations are

$$l_n = -z^{n+1}\partial_z\,, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\,, \text{ for } n \in \mathbb{Z}\,. \tag{1.1}$$

Analyse the behaviour of l_n at $z = 0$ and $z = \infty$ to deduce which generators are globally well defined. Conclude that the group of finite conformal diffeomorphisms on S^2 is generated by l_{-1}, l_0, l_1 and $\bar{l}_{-1}$, $\bar{l}_0$, $\bar{l}_1$.

b) $l_{-1} = -\partial_z$ generates the rigid translations $z \to z + b$, $b \in \mathbb{C}$ while $l_0 = -z\partial_z$ generates the complex dilatations $z \to az$, $a \in \mathbb{C}$. Go to polar coordinates $z = re^{i\varphi}$ and give the geometric interpretation of $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$.

c) Show that in complex coordinates the 2d special conformal transformation with parameter $b \equiv (-c, 0)$ acts as follows:

$$x^i \to \frac{x^i - x^2 b^i}{1 - 2b\cdot x + b^2 x^2} \quad \Longleftrightarrow \quad z \to \frac{z}{cz+1}\,. \tag{1.2}$$

It is then easy to see that $l_1 = -z^2\partial_z$ generates these special conformal transformations. All in all the globally defined conformal transformations can be written

$$z \to \frac{az+b}{cz+d}\,, \qquad a, b, c, d \in \mathbb{C}\,. \tag{1.3}$$

Invertibility enforces $ad - bc \neq 0$ and we can rescale to restrict to $ad - bc = 1$. The complex 2×2 matrix formed by a, b, c and d is thus a special linear matrix, i.e. an element of $SL(2, \mathbb{C})$. The matrix formed from $(-a, -b, -c, -d)$ gives the same transformation so that the group of conformal diffeomorphisms on S^2 is $SL(2, \mathbb{C})/\mathbb{Z}_2 \equiv PSL(2, \mathbb{C})$.

d) Consider the map

$$z \to z' = \frac{(b-c)(z-a)}{(b-a)(z-c)}\,, \quad a,b,c,d \in \mathbb{C}\,. \tag{1.4}$$

Show that this defines, up to an overall rescaling, an $SL(2,\mathbb{C})$ transformation provided a, b and c are pairwise distinct. To what points are $z = a$, $z = b$ and $z = c$ mapped to?

e) Conclude that any distinct three points on the sphere can be mapped to any other distinct three points.

2 Tree-level n-point tachyon scattering

In this exercise we explore in some detail the Gaussian integral involved in the tree-level amplitude $\mathcal{A}_n(k_1,\ldots,k_n)$ for the scattering of n closed string tachyons propagating in D dimensions. We want to show that

$$\mathcal{A}_n(k_1,\ldots,k_n) = \left\langle \prod_{j=1}^{n} :e^{ik_j\cdot X(z_j,\bar{z}_j)}: \right\rangle \propto \delta^{(D)}\left(\sum_{j=1}^{n} kj\right) \prod_{j<l} |z_j - z_l|^{\alpha' k_j\cdot k_l}\,. \tag{2.1}$$

In the lecture, you sketched the derivation for the Virasoro-Shapiro amplitude involving four tachyons.

a) Show that to evaluate the amplitude we are led to consider the following Gaussian integral

$$Z[J] \equiv \int \mathcal{D}X \exp\left[\int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\sqrt{h}\left(\frac{1}{2\pi\alpha'} X\cdot\nabla^2 X + iJ\cdot X\right)\right]\,, \tag{2.2}$$

with the source J given by

$$\sqrt{h}J^\mu(z,\bar{z}) \equiv \sum_{j=1}^{n} k_j^\mu \delta^{(2)}(z-z_j,\bar{z}-\bar{z}_j)\,. \tag{2.3}$$

Note: For this integral to make sense, we have to think to have performed a Wick rotation on spacetime also. It is the combined worldsheet Wick rotation and the spacetime one that make the oscillatory integral convergent.

b) We expand the fields into eigenmodes $X_I(z,\bar{z})$ of the Laplacian ∇^2:

$$\nabla^2 X_I(z,\bar{z}) = -\omega_I^2 X_I(z,\bar{z})\,. \tag{2.4}$$

These eigenmodes form a complete set such that we have

$$\begin{aligned} X^\mu(z,\bar{z}) &= \sum_I x_I^\mu X_I(z,\bar{z})\,, & \int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\sqrt{h} X_I(z,\bar{z}) X_{I'}(z,\bar{z}) &= \delta_{II'}\,, \\ x_I^\mu &= \int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\sqrt{h} X^\mu(z,\bar{z}) X_I(z,\bar{z})\,, & \sum_I X_I(z,\bar{z}) X_I(z',\bar{z}') &= h^{-1/2}\delta^{(2)}(z-z',\bar{z}-\bar{z}')\,. \end{aligned} \tag{2.5}$$

Use this decomposition of the fields and these properties to write

$$Z[J] = \prod_{I,\mu} \int \mathrm{d}x_I^\mu \exp\left(-\frac{w_I^2}{2\pi\alpha'} x_I^\mu x_{I,\mu} + i x_I^\mu J_{I,\mu}\right) , \tag{2.6}$$

where

$$J_I^\mu \equiv \int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\sqrt{h} J^\mu(z,\bar{z}) X_I(z,\bar{z}) . \tag{2.7}$$

c) Separate the zero mode $I = 0$ with $\omega_0 = 0$ from the non-zero modes $I \neq 0$ and evaluate the integrals to find

$$Z[J] = i(2\pi)^D \delta^{(D)}(J_0) \prod_{I\neq 0} \left(\frac{2\pi^2\alpha'}{\omega_I^2}\right)^{\frac{D}{2}} \exp\left(-\frac{\pi\alpha' J_I^\mu J_{I,\mu}}{2\omega_I^2}\right) . \tag{2.8}$$

In agreement with the note above, this is perfectly consistent to make the integral convergent by Wick rotating x_I^0.

d) Express the product as a determinant det′ which excludes zero modes and insert the definition of J_I^μ to get

$$Z[J] = i(2\pi)^D \delta^{(D)}(J_0) \det{}' \left(-\frac{\nabla^2}{2\pi^2\alpha'}\right)^{-\frac{D}{2}} \exp\left(-\frac{1}{2}\int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\sqrt{h}\mathrm{d}z'\mathrm{d}\bar{z}'\sqrt{h'} J(z,\bar{z}) \cdot J(z',\bar{z}') G'(z,\bar{z},z',\bar{z}')\right) , \tag{2.9}$$

with

$$G'(z,\bar{z},z',\bar{z}') \equiv \sum_{I\neq 0} \frac{\pi\alpha'}{\omega_I^2} X_I(z,\bar{z}) X_I(z',\bar{z}') . \tag{2.10}$$

e) Show that

$$-\frac{1}{\pi\alpha'}\nabla^2 G'(z,\bar{z},z',\bar{z}') = h^{-1/2}\delta^{(2)}(z-z',\bar{z}-\bar{z}') - X_0^2 , \tag{2.11}$$

A proper treatment of the rest of the computation requires to write the general solution to (2.11) and define a regularized Green's function to take care of the self-contractions. This is related to normal ordering (see Polchinski sects. 6.2 and 3.6 for more details). The effect of this treatment is that self-contractions are removed.

We accept that this is the case, map the sphere to the complex plane, and proceed by using

$$G'(z,\bar{z},z',\bar{z}') \sim -\frac{\alpha'}{2} \ln(|z-z'|^2) . \tag{2.12}$$

f) Bring everything together to write

$$\begin{aligned} \mathcal{A}_n(k_1,\ldots,k_n) &= i(2\pi)^D \delta^{(D)}\left(\sum_{j=1}^n k_j\right) \det{}'\left(-\frac{\nabla^2}{2\pi^2\alpha'}\right)^{-\frac{D}{2}} \prod_{j<l}^n |z_j - z_l|^{\alpha' k_j\cdot k_l} \\ &\propto \delta^{(D)}\left(\sum_{j=1}^n k_j\right) \prod_{j<l}^n |z_j - z_l|^{\alpha' k_j \cdot k_l} . \end{aligned} \tag{2.13}$$

3 ♫ Gamma gamma gamma (a man after midnight) ♫

You have seen in the lecture that the full Virasoro-Shapiro amplitude (whose piece coming from the X fields correspond to the result of the previous exercise in the case of four tachyon insertions) takes the form

$$ig_s^2 C_{S^2} (2\pi)^{26} \delta^{(26)} \left(\sum_{i=1}^{4} k_i \right) \int \mathrm{d}^2 z |z|^{-\frac{\alpha'}{2}u-4} |1-z|^{-\frac{\alpha'}{2}t-4} . \tag{3.1}$$

As in the lecture we define $C(a,b) \equiv \int \mathrm{d}^2 z |z|^{2a-2} |1-z|^{2b-2}$ and we want to show that

$$C(a,b) = \frac{2\pi \Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} , \qquad c \equiv 1-a-b . \tag{3.2}$$

a) From the definition of the Euler function

$$\Gamma(z) \equiv \int_0^{+\infty} \mathrm{d}t t^{z-1} e^{-t} , \tag{3.3}$$

show that you can write

$$|z|^{2a-2} = \frac{1}{\Gamma(1-a)} \int_0^{+\infty} \mathrm{d}t t^{-a} e^{-|z|^2 t} , \text{ and } |1-z|^{2b-2} = \frac{1}{\Gamma(1-b)} \int_0^{+\infty} \mathrm{d}u u^{-b} e^{-|1-z|^2 u} . \tag{3.4}$$

b) Insert this in $C(a,b)$, decompose the complex coordinate $z = x + iy$ and compute the integral over x and y. You should find

$$C(a,b) = \frac{2\pi}{\Gamma(1-a)\Gamma(1-b)} \int_0^{+\infty} \mathrm{d}t \mathrm{d}u \frac{t^{-a} u^{-b}}{t+u} e^{-tu/(t+u)} . \tag{3.5}$$

c) Change variables to $t \equiv \alpha\beta$ and $u \equiv (1-\beta)\alpha$ with $\alpha \in [0, +\infty)$ and $\beta \in [0,1]$. Recognize a Euler function in the integral over α and write $1-a-b = c$ to get

$$C(a,b) = \frac{2\pi\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)} \int_0^1 \mathrm{d}\beta (1-\beta)^{a-1} \beta^{b-1} \equiv \frac{2\pi\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)} B(b,a) . \tag{3.6}$$

where B is the Euler beta function.

d) Show that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. To do this follow these steps:

- Write
$$\Gamma(x)\Gamma(y) = \int_0^{+\infty} \mathrm{d}u \int_0^{+\infty} \mathrm{d}v e^{-u} u^{x-1} e^{-v} v^{y-1} , \tag{3.7}$$
change variables to $u \equiv s^2$, $v \equiv t^2$ and extend the integrals to the whole real line.
- Go to polar coordinates $s = r\cos\theta$, $t = r\sin\theta$ and recognize $\Gamma(x+y)$ and $B(x,y)$ to conclude.