String theory lecture - Exercise sheet 9

To be discussed on December 18$^{\text{th}}$

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The goal of this exercise sheet is first to compute the propagator of the free boson on the sphere. We then evaluate some important OPE's involving the energy-momentum tensor. Eventually we rederive the Virasoro algebra from the OPE's and we evaluate how the energy-momentum tensor varies under conformal transformations.

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1 Free boson propagator

The action for the free boson is

$$S[X] = T \int \mathrm{d}z\mathrm{d}\bar{z}\partial_z X(z,\bar{z})\partial_{\bar{z}} X(z,\bar{z})\,, \tag{1.1}$$

and the energy-momentum tensor if given by

$$T(z) = -\frac{1}{\alpha'} : \partial X(z)\partial X(z) : \,. \tag{1.2}$$

a) Apply the identity

$$\int_M \mathcal{D}\phi(x')\frac{\delta F[\phi(x')]}{\delta\phi(x)} = 0\,, \tag{1.3}$$

for $F[X(z',\bar{z}')] = X(z',\bar{z}')e^{-S[X(z',\bar{z}')]}$ to find the relation

$$\partial_{\bar{z}}\partial_z\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\pi\alpha'\delta^{(2)}(z-w,\bar{z}-\bar{w})\,. \tag{1.4}$$

b) Stokes theorem on the complex plane implies

$$\int_U (\partial_{\bar{z}}F - \partial_z G)\,\mathrm{d}z\mathrm{d}\bar{z} = -i\oint_{\partial U}(F\mathrm{d}z + G\mathrm{d}\bar{z})\,, \tag{1.5}$$

where $F(z,\bar{z})$ and $G(z,\bar{z})$ are continuously differentiable functions on an open region U of $\mathbb{C}$. Apply this formula with $G=0$ and $F=\frac{1}{z}$ on the ball of radius r center around the origin to find

$$2\pi\delta^{(2)}(z,\bar{z}) = \partial_{\bar{z}}\frac{1}{z}\,. \tag{1.6}$$

c) Manipulate $\partial_{\bar{z}}\frac{1}{z}$ to make a logarithm of the modulus squares appear[1] and conclude that

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\alpha'}{2}\ln(|z-w|^2)\,. \tag{1.7}$$

From this deduce $\langle\partial_z X(z)\partial_w X(w)\rangle$.

[1] Can you see why any other rewriting would not be consistent?

2 Some OPE's

In all the following questions you will have to make use of the Wick's theorem to compute the OPE's.

a) Redo carefully the computation of the OPE of $T(z)\partial X(w)$ and compare with what is expected for a primary operator to deduce the conformal weights h, $\bar{h}$ of $\partial X(w)$.

b) Now compute the OPE of $T(z)T(w)$ to find

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{finite}\,. \tag{2.1}$$

c) Compute the OPE of $\partial X(z) :e^{ikX(w,\bar{w})}:$. For this expand the exponential and you should find

$$\partial X(z) :e^{ikX(w,\bar{w})}: = -\frac{i\alpha' k}{2}\frac{:e^{ikX(w,\bar{w})}:}{z-w} + \text{finite}\,. \tag{2.2}$$

d) Compute the following OPE:

$$T(z) :e^{ikX(w,\bar{w})}: = \frac{\alpha' k^2}{4}\frac{:e^{ikX(w,\bar{w})}:}{(z-w)^2} + \frac{\partial :e^{ikX(w,\bar{w})}:}{z-w} + \text{finite}\,. \tag{2.3}$$

Conclude on the nature of the field $:e^{ikX(w,\bar{w})}:$.

3 Energy-momentum tensor

The Virasoro generators L_n, $n \in \mathbb{Z}$ on the complex plane are given by

$$L_n = \oint_C \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z)\,, \tag{3.1}$$

where C is a contour which goes around the origin.

a) Write the commutator $[L_m, L_n]$ and deform appropriately the contour of the $\mathrm{d}z$ integration for the two terms to find

$$[L_m, L_n] = \oint_C \frac{\mathrm{d}w}{2\pi i}\oint_{C_w}\frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} \mathcal{R}[T(z)T(w)]\,. \tag{3.2}$$

In this relation C_w is a contour about $z = w$ and $\mathcal{R}$ denotes radial ordering. Note that in what follows we will not write the radial ordering explicitly anymore.

b) Thanks to the OPE of $T(z)T(w)$ found in the previous exercise (replace $1/2$ by $c/2$), rederive the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,. \tag{3.3}$$

To do this, make also use of the Cauchy-Riemann formula:

$$\oint_{C_w}\frac{\mathrm{d}z'}{2\pi i}\frac{f(z')}{(z'-z)^n} = \frac{1}{(n-1)!}f^{(n-1)}(z)\,. \tag{3.4}$$

c) Use the conformal Ward-Takahashi identity

$$\delta\mathcal{O}(z) = -\oint_{C_z} \frac{\mathrm{d}z'}{2\pi i}\epsilon(z')T(z')\mathcal{O}(z)\,, \tag{3.5}$$

and the OPE of $T(z')T(z)$ to compute the following variation of the energy-momentum tensor under the conformal transformation $z \to \tilde{z} = z + \epsilon(z)$:

$$\delta T(z) = -\epsilon(z)\partial_z T(z) - 2\epsilon'(z)T(z) - \frac{c}{12}\epsilon'''(z)\,. \tag{3.6}$$

d) Check that the transformation

$$T(z) \to \tilde{T}(\tilde{z}) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}S(z,\tilde{z})\right]\,, \tag{3.7}$$

where the Schwarzian derivative is defined as

$$S(z,\tilde{z}) = \frac{\partial^3\tilde{z}}{\partial z^3}\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2\tilde{z}}{\partial z^2}\right)^{2}\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-2}\,, \tag{3.8}$$

reproduces the infinitesimal variation computed in the previous question.

e) One can show (cf BLT p72) that the finite transformation is actually uniquely fixed by the infinitesimal variation and additional consistency requirements. Consider the map $w \to z(w) = e^{-2i\pi w/l}$ from the cylinder to the plane and find the relation

$$T_{\mathrm{cylinder}}(w) = \left(\frac{2\pi}{l}\right)^2\left(-z^2 T_{\mathrm{plane}}(z) + \frac{c}{24}\right)\,. \tag{3.9}$$