String theory lecture - Exercise sheet 8

To be discussed on December 11^{th}

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The goal of this exercise sheet is first to explore the concept of *orientifold projection* to generate different gauge groups on stacks of branes. We then study the conformal group in arbitrary dimensions and focus on two dimensions. Eventually we explore properties of primary fields mentioned in the lecture.

1 Back to D-branes with orientifold

In the last tutorial we saw why we expect a stack of N coincident D-branes to carry a $U(N)$ gauge group. We saw that when we move all the branes apart, keeping them parallel, some scalars and vector states acquire a mass and the gauge group is broken to $U(1)^N$. In this exercise, let us remind ourselves why the brane separation induces a mass for a string stretched between them, and then explore the concept of *orientifolding* to generate different gauge groups.

a) The mode expansion of a string with DD boundary conditions is

$$X^\mu(\tau,\sigma) = x_0^\mu + \frac{x_1^\mu - x_0^\mu}{l}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-i\pi n\tau/l}\sin\left(\frac{\pi n\sigma}{l}\right)\,, \tag{1.1}$$

where x_0^μ and x_1^μ are the coordinates of the endpoints of the string. In this case we saw in exercise sheet 3 that we have

$$\alpha_0^\mu = \frac{\Delta x^\mu}{\pi\sqrt{2\alpha'}}\,, \qquad \text{with} \qquad \Delta x^\mu \equiv x_1^\mu - x_0^\mu\,. \tag{1.2}$$

Compute then the general form of the mass shell condition:

$$M^2 = 2p^+p^- - p^ip^i = \frac{1}{\alpha'}(N_\perp + \alpha'(T\Delta x)^2 + a_\perp) \tag{1.3}$$

b) Show that after gauge fixing, the classical worldsheet action is invariant under the parity transformation

$$\tau \to \tau\,, \qquad \sigma \to l - \sigma\,. \tag{1.4}$$

c) At the quantum level we implement this *orientifold symmetry* with a unitary operator Ω acting on the string fields as

$$X^\mu(\tau,\sigma) \to \Omega^\dagger X^\mu(\tau,\sigma)\Omega = X^\mu(\tau, l-\sigma)\,. \tag{1.5}$$

The Polyakov action is invariant under the orientifolding and so are the equations of motion and the solutions. Therefore, act on the various mode expansions of the string with the orientifolding and impose that these expansions are invariant to find the following actions on the modes:

$$\begin{aligned} &\text{Closed string:} && \Omega^\dagger \alpha_n^\mu \Omega = \tilde{\alpha}_n^\mu\,, && \Omega^\dagger \tilde{\alpha}_n^\mu \Omega = \alpha_n^\mu\,, \\ &\text{Open string NN:} && \Omega^\dagger \alpha_n^\mu \Omega = (-1)^n \alpha_n^\mu\,, && \\ &\text{Open string DD:} && \Omega^\dagger \alpha_n^\mu \Omega = (-1)^{n+1} \alpha_n^\mu\,, && \Omega^\dagger x_{0/1} \Omega = x_{1/0}\,, \\ &\text{Open string DN:} && \Omega^\dagger \alpha_{n+\frac{1}{2}}^\mu \Omega = i(-1)^{n+1} \alpha_{n+\frac{1}{2}}^\mu\,. && \end{aligned} \tag{1.6}$$

You can find these results mentioned in Blumenhagen-Lüst-Theisen p50.

d) The idea is now to define the orientifolded theory by keeping only those states in the string spectrum which are invariant under the orientifold action. What we are doing is considering the theory quotiented by the orientifold $\mathbb{Z}_2$ action. Act with Ω on the first-excited state for the closed string and conclude which fields are kept and which are projected out. To do that:

- Insert $\Omega^\dagger\Omega = 1$ where you need it.
- Use $\Omega^2 = \left(\Omega^\dagger\right)^2 = 1$ to write
$$\Omega \alpha_n^i \Omega^\dagger = \Omega^\dagger \alpha_n^i \Omega\,. \tag{1.7}$$
- Use the fact that the vacuum is even under the orientifold (we cannot prove this now but it comes from the so-called *tadpole cancellation condition*).
- Use the result of the previous question to show what is the effect of the projection on the polarization vector.
- Deduce what fields survive.

e) Apply the same procedure to find the effect of the orientifolding on strings with both ends attached to a single D-brane. Do this in the two cases where the vacuum is even or odd.

f) Consider a stack of N D-branes. Depending on the vacuum being even or odd, the orientifold action on the vacuum is
$$\Omega\,|0,p,r,s\rangle = \pm\,|0,p,s,r\rangle\ , \tag{1.8}$$
where you see that the Chan-Patton indices are exchanged. What does this mean for the vector states kept at the first excited level for the two signs? How many states are kept?

g) If you would interpret these states as coming from an adjoint representation, what groups would they correspond to?

h) What one-loop worldsheet topologies do you expect to encounter for oriented closed and open strings? What about non-oriented strings?

2 The conformal group

At the infinitesimal level and for arbitrary number of spacetime dimensions D, a conformal transformation $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ satisfies

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = 2\omega(x)\eta_{\mu\nu} \,. \tag{2.1}$$

a) From this relation, show that we have

$$\begin{aligned} \omega(x) &= \frac{1}{D}\partial^{\mu}\epsilon_{\mu} \,, \\ (\eta_{\mu\nu}\partial^2 + (D-2)\partial_{\mu}\partial_{\nu})\omega(w) &= 0 \,, \\ (D-1)\partial^2\omega(x) &= 0 \,. \end{aligned} \tag{2.2}$$

To do this you will have to:

- Take the trace of the relation,
- Take the divergence,
- Contract it further with either ∂^{μ} or ∂^{ν}.

b) Explain again what is so special about the $D = 2$ case.

c) The conformal transformations in two dimensions are generated by

$$l_n \equiv -z^{n+1}\partial_z \,, \qquad \bar{l}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}} \,. \tag{2.3}$$

Verify that these generators indeed satisfy Witt algebras:

$$[l_m, l_n] = (m-n)l_{m+n} \,, \qquad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \,, \qquad [l_m, \bar{l}_n] = 0 \,. \tag{2.4}$$

3 Primary fields and radial quantization

By definition a primary field $\mathcal{O}(z,\bar{z})$ with weights h, $\bar{h}$ transforms under a conformal transformation $z \to z'$, $\bar{z} \to \bar{z}'$ as

$$\mathcal{O}(z,\bar{z}) \to \mathcal{O}'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \mathcal{O}(z,\bar{z}) \,. \tag{3.1}$$

a) Compute how a primary field with conformal weights h, $\bar{h}$ transforms under dilatations $z \to e^{\lambda}z$ and rotations $z \to e^{i\theta}z$ with λ, $\theta \in \mathbb{R}$. Deduce a physical interpretation for $h + \bar{h}$ and $h - \bar{h}$.

b) For an infinitesimal conformal transformation

$$z'(z) = z + \epsilon(z) \qquad \bar{z}'(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z}) \,, \tag{3.2}$$

show that the variation of a primary field $\delta\mathcal{O}(z,\bar{z})$ is given by

$$\delta\mathcal{O}(z,\bar{z}) = -\left(h\partial_z\epsilon + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \epsilon\partial_z + \bar{\epsilon}\partial_{\bar{z}}\right)\mathcal{O}(z,\bar{z}) \,. \tag{3.3}$$

c) As you have seen in the lecture the variation of an operator is given by the conformal Ward-Takahashi identity

$$\delta\mathcal{O}(w,\bar{w}) = -\frac{1}{2\pi i}\oint \mathrm{d}z\,\Big(\epsilon(z)T(z) + \bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\Big)\,\mathcal{O}(w,\bar{w})\,, \tag{3.4}$$

where the contour is counter-clockwise in z and $\bar{z}$. It can be rewritten like

$$\delta\mathcal{O}(w,\bar{w}) = -\mathrm{Res}_{z-w}\left[\epsilon(z)T(z)O(w,\bar{w})\right] - \mathrm{Res}_{\bar{z}-\bar{w}}\left[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})O(w,\bar{w})\right]\,. \tag{3.5}$$

Compare this result to the transformation rule of a primary operator uncovered in the previous question to find the OPE between T and $\mathcal{O}$:

$$\begin{aligned} T(z)\mathcal{O}(w,\bar{w}) &= h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \text{finite}\,,\\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \bar{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \text{finite}\,. \end{aligned} \tag{3.6}$$

d) The cylinder is mapped to the Riemann sphere thanks to the projection

$$w \to z = -e^{-i\frac{2\pi}{l}w}\,. \tag{3.7}$$

Argue that time-ordering on the cylinder correspond to radial-ordering on the Riemann sphere.

e) Show that for a primary field $\phi = \phi_{\mathrm{L}}(\sigma^-) + \phi_{\mathrm{R}}(\sigma^+)$ a mode expansion

$$\phi_{\mathrm{L}}(\sigma^-) = \left(-\frac{2\pi i}{l}\right)^h \sum_n \phi_n e^{-\frac{2\pi i}{l}n\sigma^-}\,, \qquad \phi_{\mathrm{R}}(\sigma^+) = \left(-\frac{2\pi i}{l}\right)^{\bar{h}} \sum_n \tilde{\phi}_n e^{-\frac{2\pi i}{l}n\sigma^+}\,, \tag{3.8}$$

translates into $\phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ with

$$\phi(z) = \sum_n \phi_n z^{-n-h}\,, \qquad \bar{\phi}(z) = \sum_n \phi_n \bar{z}^{-n-\bar{h}}\,. \tag{3.9}$$