# String theory lecture - Exercise sheet 7

To be discussed on December $4^{\text{th}}$

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The goal of this exercise sheet is to study some details of the $bc$-ghost system and of the BRST formalism. We first recall how to use Berezin integration to evaluate determinants. We then analyse the effect of the $bc$-ghost ordering constant on the total system. Eventually, we consider the generic Faddeev-Popov procedure to demonstrate the BRST invariance, check how the particular case of the bosonic string emerges, and we redo the constraint analysis on physical states of the open string performed in the lecture.

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## 1 Grassmann determinant

Grassmann variables $\theta_i$, $i \in \{1, \dots, n\}$ are anti-commuting:

$$\theta_i \theta_j = -\theta_j \theta_i \qquad \Longrightarrow \qquad \theta_i^2 = 0 \,. \tag{1.1}$$

The Berezin integration over Grassmann variables is defined such that

$$\int \mathrm{d}\theta_i = 0 \,, \qquad\qquad \int \mathrm{d}\theta_i \theta_j = \delta_{ij} \,. \tag{1.2}$$

a) From the properties of the Berezin integration you can deduce

$$\int \mathrm{d}^n\theta \theta_{i_1} \dots \theta_{i_n} = \epsilon_{i_1 \dots i_n} \,. \tag{1.3}$$

Use the Taylor expansion of the exponential to show that for a $n \times n$ matrix $M$, one has

$$\int \prod_{i=1}^{n} \mathrm{d}\psi_i \mathrm{d}\theta_i \exp\left(\theta^T M \psi\right) = \det M \,. \tag{1.4}$$

## 2 Ghosts and Casimir

The $bc$-ghost system is quantized such that

$$\{b_m, c_n\} = \delta_{m+n} \,, \quad \{b_m, b_n\} = \{c_m, c_n\} = 0 \,, \tag{2.1}$$

and their energy-momentum tensor modes are

$$L_m^{\mathrm{g}} = \sum_{n \in \mathbb{Z}} (m-n) : b_{m+n} c_{-n} : \,. \tag{2.2}$$

a) Use $\zeta$-regularization to compute the normal-ordering constant $a^{\mathrm{g}}$ in $L_0^{\mathrm{g}}$.

b) In total we have $L_m = L_m^X + L_m^{\mathrm{g}} + a\delta_m$.

- What is the ordering constant of the full system containing $D$ $X$-bosons and a $bc$-ghost?
- Compare the result with the one obtained in light-cone gauge quantization.
- What is the effect of the $bc$-ghost?

c) One can show that the ghosts satisfy the following Virasoro algebra:

$$[L_m^{\mathrm{g}}, L_n^{\mathrm{g}}] = (m-n)L_{m+n}^{\mathrm{g}} + \frac{1}{6}(m - 13m^3)\delta_{m+n}\,. \tag{2.3}$$

Write the commutation relation of the combined Virasoro generators $[L_m, L_n]$ and explain what is the effect of the criticality condition.

d) In the Faddeev-Popov treatment of the path integral, a central extension in the Virasoro algebra signals a Weyl anomaly. Reinterpret the criticality condition. Is it fundamentally the number of spacetime dimensions that is constrained?

# 3 BRST quantization

Recall from the lecture that we consider an action $S_\phi[\phi]$ invariant under a gauge symmetry with parameter $\Delta^\alpha$

$$\phi \to \phi + \Delta^\alpha \delta_\alpha \phi\,, \tag{3.1}$$

where the transformation operators form a Lie algebra with structure constants $f_{\alpha\beta}{}^\gamma$:

$$[\delta_\alpha, \delta_\beta] = f_{\alpha\beta}{}^\gamma \delta_\gamma\,. \tag{3.2}$$

In addition we fix the gauge by imposing conditions, labelled by an index $A$, of the form

$$F^A[\phi] = 0\,. \tag{3.3}$$

The Faddeev-Popov quantization of this system leads to a path integral

$$Z = \int \mathcal{D}\phi \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha \exp\left(-S_\phi[\phi] - S_{\mathrm{gf}}[B,\phi] - S_{\mathrm{FP}}[b,c,\phi]\right)\,, \tag{3.4}$$

where the gauge-fixing and FP actions read

$$S_{\mathrm{gf}}[B,\phi] = \int -iB_A F^A[\phi]\,, \qquad S_{\mathrm{FP}}[b,c,\phi] = \int b_A c^\alpha \delta_\alpha F^A[\phi]\,. \tag{3.5}$$

a) The full action is invariant under the global fermionic BRST symmetry acting like

$$\delta_\epsilon \phi = -i\epsilon c^\alpha \delta_\alpha \phi\,, \quad \delta_\epsilon B_A = 0\,, \quad \delta_\epsilon b_A = \epsilon B_A\,, \quad \delta_\epsilon c^\alpha = \frac{i}{2}\epsilon c^\beta c^\gamma f_{\beta\gamma}{}^\alpha\,. \tag{3.6}$$

Our goal is to check this invariance. To do that follow these steps:

- Argue why $\delta_\epsilon S_\phi = 0$.
- Show that $\delta_\epsilon(b_A F^A) = i\epsilon(S_{\text{gf}} + S_{\text{FP}})$.
- Show that $\delta_\epsilon \delta_{\epsilon'} \Phi = 0$ for $\Phi \in \{\phi, B_A, b_A, c^\alpha\}$.
- Show that $\delta_\epsilon \delta_{\epsilon'}(\Phi\Psi) = 0$ for $\Phi, \Psi \in \{\phi, B_A, b_A, c^\alpha\}$ and deduce that $\delta_\epsilon \delta_{\epsilon'} G(\Phi) = 0$ for any function $G(\Phi)$ and $\Phi \in \{\phi, B_A, b_A, c^\alpha\}$.
- Conclude that the full action is BRST invariant.

b) The bosonic string is a special case of the general formalism upon choosing

$$\begin{aligned} &\phi \leftrightarrow \{X^\mu(\tau,\sigma), h_{ab}(\tau,\sigma)\}\,, && S_\phi \leftrightarrow S_{\text{P}}\,, && c^\alpha \leftrightarrow \{c^a, c\}\,, \\ &b_A \leftrightarrow b_{ab} + \frac{1}{2} b h_{ab}\,, && F^A \leftrightarrow \frac{\sqrt{-h}}{4\pi}(h^{ab} - \hat{h}^{ab})\,, && \end{aligned} \tag{3.7}$$

where $c^a$ and $c$ are respectively the diffeomorphisms and Weyl ghosts, and the symmetric tensor $b_A$ is decomposed into a traceless part $b_{ab}$ and a trace part $\frac{1}{2} b h_{ab}$.

Show that with these identifications the FP action becomes

$$S_{\text{FP}}^{\text{string}} = \int \mathrm{d}^2\xi \sqrt{-\hat{h}}\, b^{ab} (\hat{P} \cdot c)_{ab}\,. \tag{3.8}$$

For that follow these steps:

- Write down explicitly the variations $\delta_{c^a} F^{ab}$ and $\delta_c F^{ab}$. Do not bother writing in detail the terms proportional to $(h^{ab} - \hat{h}^{ab})$ since they will disappear after integration over $B_{ab}$.
- Perform the path integral over $b$ and $c$ by expanding $b(\sigma)$, $c(\sigma)$ and $(\nabla \cdot c)(\sigma)$ into a sum of complete functions (orthogonal to each other):

$$b(\sigma) \equiv \sum_n b_n Y_n(\sigma)\,, \qquad c(\sigma) \equiv \sum_n c_n Y_n(\sigma)\,, \qquad (\nabla \cdot c)(\sigma) \equiv \sum_n (\nabla \cdot c)_n Y_n(\sigma)\,. \tag{3.9}$$

  Note: This derivation of the FP action for the string is equivalent to the one you have seen in the lecture. There you have first manipulated the inverse FP determinant to integrate out $\omega$ and only after this, you have introduced the diffeomorphism ghosts. Here in the exercise we have first introduced all the ghosts (diffeomorphism + Weyl) and then integrated over the Weyl ones which disappear along with the $\nabla \cdot c$ term.

c) The BRST charge of the open string is

$$Q = \frac{1}{2} \sum_{n,m \in \mathbb{Z}} c_n : \alpha_m \cdot \alpha_{-n-m} : + \frac{1}{2} \sum_{n,m \in \mathbb{Z}} (n-m) : c_n c_m b_{-n-m} : - c_0\,. \tag{3.10}$$

Compute carefully $Q\,|0, p, \downarrow\rangle$ to find the mass-shell condition displayed in the lecture.

d) Consider the first-excited state

$$|\psi\rangle = (\epsilon \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})\,|0, p, \downarrow\rangle\,. \tag{3.11}$$

At this point this state contains $26 + 2$ degrees of freedom. Redo the computation outlined in the lecture to find that $Q\,|\psi\rangle = 0 \Rightarrow p \cdot \epsilon = 0$ and $\beta = 0$. This reduces the number of degrees of freedom to 26.

e) Show that, in the BRST cohomology, one has the equivalence $\epsilon \sim \epsilon + \beta' p$, with $\beta' \in \mathbb{C}$. This brings us down to 24 degrees of freedom, as required.