String theory lecture - Exercise sheet 6

To be discussed on November 27$^{\text{th}}$

Lecturer: Prof. Arthur Hebecker Head tutor: Dr. Thibaut Coudarchet

The goal of this exercise sheet is first to compute the ordering constants of the open string for all possible boundary conditions in lightcone gauge. With this, we can then analyse the spectrum arising from a particular brane system. Eventually, we go through the Faddeev-Popov (FP) procedure applied to the electromagnetic field to remind ourselves of this strategy.

1 ζ-regularization

The generalized ζ-function (also called Hurwitz function) is defined for $\mathrm{Re}\,(s) > 1$ as follows:

$$\zeta(s,q) \equiv \sum_{n=0}^{+\infty} (n+q)^{-s}\,. \tag{1.1}$$

It can be analytically continued to $\mathrm{Re}\,(s) < 1$ with the representation

$$\zeta(s,q) = \frac{1}{s-1}\sum_{n=0}^{+\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(q+k)^{1-s}\,. \tag{1.2}$$

When $s < 1$ is an integer, the function becomes

$$\zeta(s,q) = \frac{1}{s-1}B_{1-s}(q)\,, \tag{1.3}$$

where $B_n(q)$ denotes the Bernouilli polynomials.

a) Use the definition of the Bernouilli polynomials:

$$\frac{te^{qt}}{e^t-1} \equiv \sum_{n=0}^{\infty} B_n(q)\frac{t^n}{n!}\,, \tag{1.4}$$

to compute $B_2(q)$. Express then $\zeta(-1,q)$ and in particular give the value of $\zeta(-1,1)$.

b) Use the result above to compute the ordering constant of an open string coming from a single dimension $i \in \{2,\ldots,D-1\}$ with either NN/DD or ND/DN boundary conditions. To do this remember the form of the mode decompositions that we have computed in exercise sheet 3.

c) Come up with another derivation of the ordering constant for a direction with ND/DN boundary conditions that only uses the regular ζ-function $\zeta(-1,1)$. *Hint:* Write the sum over half-integers in terms of sums over odd and even integers.

d) Assuming in full generality that the string has m DN/ND directions and $(D-2)$ NN/DD directions, compute the total ordering constant and derive the mass of the string.

2 String spectrum on D-branes

Consider the following setup of D-branes (all present at the same time) for the critical bosonic string:

- One spacetime-filling D25-brane,
- A stack of $N > 1$ D12-branes along direction $\mu \in \{0, 1, \ldots, 12\}$.

Our goal is to describe the open-string spectrum of this brane system up to the first excited level. You can refer to the book by Blumenhagen, Lüst, Theisen (BLT), *Basic concepts of string theory*, section 3.3, to help you navigate this exercise.

a) We start with the D25-D25 sector. This means that we have NN boundary conditions on all directions and this matches the spectrum that you have seen in the lecture. Rederive this spectrum (give explicit states + masses) up to first excited level.

b) Now we look at the D12-D12 sector. How do you think we should take care of the fact that we are dealing with a stack of coincident branes in order to label the states? You need to introduce new quantum numbers called the *Chan-Patton* factors. What is then the form of an arbitrary state?

c) Describe the spectrum up to level one by splitting excitations along the stack of branes and transverse to it. How do you think we interpret these excitations (see BLT)?

d) The only sector remaining is the D12-D25 sector. Using the results of the previous exercise, what is the total ordering constant in this sector and what does this imply?

e) Describe the spectrum in this sector up to level one.

3 Faddeev-Popov in QED

We start with the partition function

$$Z = \int \mathcal{D}A e^{iS[A]}\,, \tag{3.1}$$

where $S[A]$ is the action for QED.

a) Apply the same Faddeev-Popov strategy outlined in the lecture to take care of the gauge symmetry under

$$A_\mu(x) \to A_\mu^\alpha(x) = A_\mu(x) + \partial_\mu \alpha(x)\,. \tag{3.2}$$

To do this, follow these steps:

- Introduce a function $G(A^\alpha)$ for gauge fixing and insert in Z a generalisation of

$$1 = \int \mathrm{d}x \delta(f(x))|f'(x)|\,, \tag{3.3}$$

for the case at hand.

- Interpret the resulting path integral.
- Perform the change of variable $A \to A^\alpha$.
- You can write the determinant like

$$\left|\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\right| = \left|\det\left(\frac{\delta G(A^{\alpha+\alpha'})}{\delta\alpha'}\right)\right|_{\alpha'=0} . \tag{3.4}$$

  Rename $A^\alpha \to A$, realize that you can drop the α path integral and finally rename $\alpha' \to \alpha$ to get

$$Z = \int \mathcal{D}\mathcal{A}\delta[G(A)] \left|\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\right|_{\alpha=0} e^{iS[A]} . \tag{3.5}$$

b) What is $G(A)$ for the Lorentz gauge?

c) Show that $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$ is independent of A and can thus be taken out of the path integral.

Note: In the non-Abelian case, e.g. $SU(N)$, the FP determinant is not independent of A, i.e. the ghosts do not decouple. This situation is more in line with what we have when we quantize the string. Another crucial difference when we deal with the strings is the infinite-dimensional nature of the Virasoro algebra.

d) Consider the generalisation $G(A) = \partial_\mu A^\mu(x) - \omega(x)$ for fixing the gauge and, because we know the result cannot depend on ω, introduce a further path integral with Gaussian weight:

$$\int \mathcal{D}\omega \exp\left[-i\int \mathrm{d}^4x \frac{\omega^2}{2\xi}\right] \cdots \tag{3.6}$$

Perform the integral and show that, effectively, we have added a term $-(\partial^\mu A_\mu)^2/(2\xi)$ to the Lagrangian.