String theory lecture - Exercise sheet 4

To be discussed on November 13th

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The goal of this exercise sheet is first to develop the argument leading to the condition $D \leq 26$ in order to avoid ghosts in the spectrum of the bosonic string. We then derive usual infinitesimal variations and eventually we redo carefully the manipulations done in the lecture to show how the lightcone spacetime coordinates are fully determined in terms of the transverse ones in lightcone gauge.

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1 The ghost in the machine

a) Write the general expression of a level-two excited state for the open string.

b) Consider the particular state $|\phi\rangle$ given by

$$|\phi\rangle = \Big(c_1\alpha_{-1}\cdot\alpha_{-1} + c_2 p\cdot\alpha_{-2} + c_3(p\cdot\alpha_{-1})^2\Big)\,|0,p\rangle\ . \tag{1.1}$$

Find relations between c_1, c_2 and c_3. To do this you can follow these steps:

- Show that $[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}$.
- Compute $L_m\,|\phi\rangle$ by making use of the previous formula.
- Use the constraints $(L_0-1)\,|\phi\rangle = L_1\,|\phi\rangle = L_2\,|\phi\rangle = 0$ to find the desired relations.

c) Compute carefully the norm of the state to find

$$\langle\phi|\phi\rangle = \frac{2c_1^2}{25}(D-1)(26-D)\,, \tag{1.2}$$

and conclude.

2 Infinitesimal diffeomorphisms

Consider the following change of coordinates:

$$\xi^a \to \xi'^a = \xi'^a(\xi)\,. \tag{2.1}$$

The transformation matrices associated to this transformation are $P^a{}_{b'} \equiv \frac{\partial\xi^a}{\partial\xi^{b'}}$ and its inverse $P^{a'}{}_b \equiv \frac{\partial\xi^{a'}}{\partial\xi^b}$.

a) At the infinitesimal level, the change of coordinate reads

$$\xi'^a = \xi^a + \epsilon^a(\xi)\,. \tag{2.2}$$

Derive the following infinitesimal variations for a scalar field $\Phi(\xi)$, the metric $h_{ab}(\xi)$ and its associated density $\sqrt{-h}$:

$$\begin{aligned} \delta\Phi &= -\epsilon^a \partial_a \Phi\,, \\ \delta h_{ab} &= -\epsilon^c \partial_c h_{ab} - (h_{cb}\partial_a \epsilon^c + h_{ad}\partial_b \epsilon^d) = -(\nabla_a \epsilon_b + \nabla_b \epsilon_a)\,, \\ \delta\sqrt{-h} &= -\partial_a(\epsilon^a \sqrt{-h})\,, \end{aligned} \tag{2.3}$$

where ∇_a denotes the covariant derivative.

Comment not needed for solution: One can understand the specific and simple variation of the metric from the fact that it is covariantly constant. Indeed, we expect the variation to only come from the basis transformation on the tangent bundle, and we expect the contribution from "going from one point to another" to drop out. Another way to formulate this is to write the variation as a Lie derivative:

$$\delta h_{ab} = \mathcal{L}_\epsilon h_{ab} = -\nabla_\epsilon h_{ab} - h_{cb}\nabla_a \epsilon^c - h_{ac}\nabla_b \epsilon^c\,, \tag{2.4}$$

where the first term vanishes due to covariant constancy.

b) You can see that the quantity $\sqrt{-h}$ does not transform exactly like a scalar. One can define a *tensor density of weight* w as an object which transforms like a tensor field but with an additional J^w factor, where $J \equiv \det(P^a{}_{b'})$. For example, a tensor density $T^a{}_b$ with weight w transforms as follows

$$T^{a'}{}_{b'} = J^w P^{a'}{}_a P^b{}_{b'} T^a{}_b\,, \tag{2.5}$$

and the generalisation to arbitrary index structures is straightforward[1]. Given a tensor S_{ab}, show that $\sqrt{\det S_{ab}}$ is a scalar density or weight 1.

3 Lightcone gauge identities

a) By making use of the lightcone coordinates on the worldsheet, show that the constraint $T_{ab} = 0$ translates to $T_{++} = T_{--} = 0$. Show that this leads to the constraints

$$(\dot{X} \pm X')^2 = 0\,. \tag{3.1}$$

b) Write this constraint in lightcone gauge to find

$$\partial_\pm X^- = \frac{1}{2\pi\alpha' p^+}(\partial_\pm X_\perp)^2\,, \tag{3.2}$$

where $\perp$ denotes the spacetime dimensions transverse to the lightcone ones.

[1] This definition can be made coordinate-independent by introducing a *density bundle* over the manifold and by considering sections of the tensor product of tangent and cotangent bundles as usual, further tensored with this density bundle. However integration makes sense in a coordinate-independent manner only for weight-one densities like the volume density.

c) Use this result to show that for an open string with Neumann-Neumann boundary conditions one can express the X^- oscillators in terms of the transverse ones:

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha' p^+}} \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{i=2}^{D-1} \alpha_{n-m}^i \alpha_m^i \,. \tag{3.3}$$