String theory lecture - Exercise sheet 2

To be discussed on October 30$^{\text{th}}$

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The goal of this exercise sheet is to manipulate the Polyakov action for strings and to start playing with the quantized theory.

1 Polyakov action

The Polyakov action for the bosonic string is

$$S_{\mathrm{p}} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\xi \sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,. \tag{1.1}$$

a) Check that the action is invariant under

- diffeomorphisms $\xi^a \to \xi'^a(\xi^0, \xi^1)$,
- Poincaré transformations $X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + V^\mu$ with $\Lambda \in SO(1, D-1)$,
- Weyl rescalings $h_{ab}(\xi) \to h'_{ab} = \phi(\xi)h_{ab}(\xi)$.

b) For a matrix A such that $\det(A) \neq 0$, show that we have the relation

$$\delta(\det A) = (\det A)\mathrm{Tr}\left(A^{-1}\delta A\right), \tag{1.2}$$

by following two methods:

- Using the identity $\ln\det A = \mathrm{Tr}\,\ln A$,
- Using the explicit formula for $\det A$ in terms of the Levi-Civita symbol.

c) Go carefully through the derivation of the stress-energy tensor, as presented in the lecture, to find

$$T_{ab} \equiv \frac{4\pi}{\sqrt{-h}}\frac{\delta S_p}{\delta h^{ab}} = -2\pi T\left(G_{ab} - \frac{1}{2}h_{ab}G_{cd}h^{cd}\right), \tag{1.3}$$

where $G_{ab} \equiv \partial_a X^\mu \partial_b X_\mu$.

d) The equations of motion then tell you that $T^{ab} = 0$. Show that the trace $T_a{}^a$ vanishes even before imposing the equations of motion. Derive again $T_a{}^a = 0$ without even using the explicit form of T_{ab} but using instead one of the symmetries of the action.

2 Flat gauge

a) Given the symmetries of the Riemann tensor, show that of the 2^4 components of R_{abcd} in two dimensions, only one is independent.

b) Check that the ansatz $R_{abcd} = \lambda(h_{ac}h_{bd} - h_{ad}h_{bc})$ is consistent with the symmetries of the Riemann tensor. Show that $\lambda = \mathcal{R}/2$ with $\mathcal{R}$ the Ricci scalar.

c) Compute the Einstein tensor and interpret the result.

Note: As you saw in the lecture, one can show that under a Weyl rescaling, $h_{ab} \to h'_{ab} = e^{2\omega(\xi^0,\xi^1)}h_{ab}$ we have the following transformation:

$$\sqrt{-h'}\mathcal{R}' \to \sqrt{-h}(\mathcal{R} - 2\nabla^2\omega)\,. \tag{2.1}$$

This is what allows to go to a gauge where the metric has a zero Ricci scalar everywhere. From the structure of the Riemann tensor in two dimensions, this makes it possible to go to the flat gauge where $h_{ab} = \eta_{ab}$.

3 Mode expansion

The mode expansion for the closed string of length l that you have seen in the lecture is given by

$$X^\mu = x^\mu + \frac{\pi\alpha'}{l}p^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-2i\pi n\sigma^+/l} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-2i\pi n\sigma^-/l}\,, \tag{3.1}$$

with $\sigma^\pm \equiv \tau \pm \sigma$.

a) Derive the commutation relations of x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ displayed in the lectures from the canonical commutation relations of X^μ and Π^μ. To do this, use the relation

$$\int_0^l \mathrm{d}\sigma \int_0^l \mathrm{d}\sigma' e^{2\pi i m\sigma/l}e^{2\pi i n\sigma'/l}\delta(\sigma - \sigma') = l\delta_{m+n}\,. \tag{3.2}$$

b) Derive the classical expression for the Hamiltonian

$$H = \frac{\pi}{l}\sum_{n\,=-\infty}^{\infty}\alpha_{-n}\cdot\alpha_n + \frac{\pi}{l}\sum_{n\,=-\infty}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\,, \tag{3.3}$$

from the Polyakov action (do not worry about ordering issues). Here $\alpha_{-n}\cdot\alpha_n \equiv \alpha_{-n}^\mu\alpha_n^\nu\eta_{\mu\nu}$.