String theory lecture - Exercise sheet 13

To be discussed on January 29th

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The goal of this exercise sheet is to first manipulate the RNS superstring action in light-cone coordinates to check its SUSY invariance and make explicit the possibility to have periodic or anti-periodic boundary conditions for the fermions. In the second exercise, we study the circle compactification of the bosonic string to discover new stringy effects, anticipate the existence of *T-duality* and exhibit the phenomenon of *gauge enhancement*.

1 RNS superstring action

The superstring action in flat gauge takes the form

$$S = -\frac{1}{4\pi}\int \mathrm{d}^2\xi \left(\frac{1}{\alpha'}\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu\gamma^a\partial_a\psi_\mu\right) . \tag{1.1}$$

a) Using the spinor component notation $\psi \equiv (\psi_-, \psi_+)^T$, show that in light-cone coordinates the action takes the form you have seen in the lecture (use the same conventions defined there for the Clifford algebra and the gamma matrices):

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\xi \left(\frac{2}{\alpha'}\partial_+ X^\mu \partial_- X_\mu + i(\psi_-^\mu\partial_+\psi_{-\mu} + \psi_+^\mu\partial_-\psi_{+\mu})\right) . \tag{1.2}$$

b) The on-shell SUSY transformations are given by

$$\sqrt{\frac{2}{\alpha'}}\delta X^\mu = i\bar{\epsilon}\psi^\mu , \qquad \delta\psi^\mu = \sqrt{\frac{2}{\alpha'}}\frac{1}{2}\gamma^a\partial_a X^\mu\epsilon , \tag{1.3}$$

with the Majorana spinor $\epsilon \equiv (\epsilon_-, \epsilon_+)^T$ subject to the chiral condition

$$\gamma^b\gamma_a\partial_b\epsilon = 0 . \tag{1.4}$$

Fix $\alpha' = 2$ for convenience and show that in light-cone coordinates these transformations become

$$\delta X^\mu = i(\epsilon_+\psi_- - \epsilon_-\psi_+) , \qquad \delta\psi_\pm^\mu = \pm\epsilon_\mp\partial_\pm X^\mu , \tag{1.5}$$

with the condition

$$\partial_+\epsilon_+ = \partial_-\epsilon_- = 0 . \tag{1.6}$$

c) It is convenient to define $\eta^- \equiv \epsilon_+$ and $\eta^+ = -\epsilon_-$. Rewrite the transformations with this parameter. The transformations parametrized by η^+ and η^- are completely independent so that on-shell invariance of the action can be checked separately for both of them. Consider then the η^+ transformation and prove the invariance of the action, on-shell.

d) Use the Noether trick by replacing η^+ by $\rho\eta^+$ and deduce from the variation of the action the following supercurrent component:

$$j_+ = \eta^+ \partial_+ X^\mu \psi_{+\mu} \,, \tag{1.7}$$

which satisfies $\partial_- j_+ = 0$.

e) Vary the action of $\psi^\mu_\pm$ to show explicitly why periodic and anti-periodic boundary conditions are allowed (you may assume that $\delta\psi^\mu_\pm|_{\tau=\pm\infty} = 0$).

2 Circle compactification

Consider the theory of a massless scalar field $\phi(x^M)$ in $d+1$ spacetime dimensions. We compactify the $(d+1)^{\text{th}}$ dimension on a circle of radius R. It means we identify $x^d \cong x^d + 2\pi R$ and our spacetime geometry is $\mathbb{R}^{1,d-1} \times S^1(R)$. Splitting the spacetime index like $M \equiv \{\mu, d\}$ with $\mu = 0, \ldots, d-1$, we can exand the field along the compact direction like

$$\phi(x^M) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) \exp\left(\frac{inx^d}{R}\right) . \tag{2.1}$$

a) What are the eigenvalues of the momentum operator in the compact direction?

b) Starting from the equation of motion in $d+1$ dimensions, show that from the effective d-dimensional point of view the modes $\phi_n(x^\mu)$ form an infinite tower of fields with mass-squared $m^2 = -p_\mu p^\mu = \frac{n^2}{R^2}$. These modes are called the *Kaluza-Klein (KK) modes* and n is the *KK number*.

c) Now we turn to the string with a target space featuring one compact dimension, such that $X^d \cong X^d + 2\pi R$ along that direction (you can think of it as being the 25^{th} direction in the bosonic string for example). Argue that the periodicity condition for the closed string (of length 2π) can then be generalized like

$$X^d(\sigma + 2\pi) = X^d(\sigma) + 2\pi R w \,, \qquad w \in \mathbb{Z} \,. \tag{2.2}$$

The new integer w is called *winding number*. Give a geometric interpretation of this number (you can draw a picture).

d) Recall what is the most general solution of $\partial_+\partial_- X^d(\sigma) = 0$ for the closed string by introducing arbitrary left and right zero modes α_0^d and $\tilde{\alpha}_0^d$ (focus on the non-oscillator part and do not impose any periodicity yet).

e) Impose the periodicity condition (2.2) to find a relation between α_0^d and $\tilde{\alpha}_0^d$.

f) Use your knowledge about the KK modes to constrain the center or mass momentum given by $(\alpha_0^d + \tilde{\alpha}_0^d)/\sqrt{2\alpha'}$. Now you can fully express α_0^d and $\tilde{\alpha}_0^d$ as well as the mode expansion in terms of the KK and winding numbers.

g) From the Virasoro constraints $(L_0 - 1)\,|\text{phys}\rangle = 0$ and $(\tilde{L}_0 - 1)\,|\text{phys}\rangle = 0$, derive the following level-matching condition and effective mass-squared in d dimensions

$$
\begin{aligned}
N - \tilde{N} &= nw\,, \\
\alpha' m^2 &= \alpha' \left[\frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2}\right] + 2N + 2\tilde{N} - 4\,.
\end{aligned}
\tag{2.3}
$$

h) What happens under the transformation

$$
n \leftrightarrow w\,, \qquad R \to R' = \frac{\alpha'}{R}\;? \tag{2.4}
$$

Try to gain physical understanding of the situation and derive the value of the radius which is a fixed point of the transformation. It is called *self-dual radius.*

i) Let us now look at the spectrum. The ground state with $n = w = 0$ and no oscillator gives the usual tachyon with $m^2 = -4/\alpha'$. At the massless level with $N = \tilde{N} = 1$ and $n = w = 0$, write the possible states by acting either with α^μ_{-1}, $\mu = 0, \ldots, d-1$ or with α^d_{-1} (and their tilde counterparts). You should find the d-dimensional graviton as well as two d-dimensional vectors and one scalar. Interpret this.

j) Now we consider the sector with $n = w = \pm 1$. The level-matching condition is modified to $N = \tilde{N} + 1$. Write the spectrum obtained with $\tilde{N} = 0$ and $N = 1$. You should find two vectors and two scalars, one of each for $n = w = 1$ and the other for $n = w = -1$. Express the mass of these states.

k) You would find another copy of these states by looking at the sector with $n = -w = \pm 1$. What happens to their mass at the self-dual radius defined above? Interpret again.