String theory lecture - Exercise sheet 12

To be discussed on January 22$^{\text{nd}}$

Lecturer: Prof. Arthur Hebecker — Head tutor: Dr. Thibaut Coudarchet

The goal of this exercise sheet is to first investigate the various spinor representations and which one are allowed depending on the dimension. In the second exercise we perform basic superspace-related computations while in the third one we derive the superstring action and the global supersymmetry transformations in components.

1 Clifford algebra and its representations

The Clifford algebra in d dimensions is defined by

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \,. \tag{1.1}$$

Note that we use a sign convention different from the lecture. These gamma matrices defined here are thus related to the one of the lecture by a factor i: $\Gamma^\mu = i\gamma^\mu$.

(a) Show that $\Sigma^{\mu\nu} \equiv -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$ satisfy the $SO(1, d-1)$ algebra:

$$i[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\sigma}\Sigma^{\nu\rho} \,. \tag{1.2}$$

(b) We assume that $d = 2k + 2$ and we define

$$\Gamma^{0\pm} \equiv \frac{1}{2}(\pm\Gamma^0 + \Gamma^1) \,, \qquad \Gamma^{a\pm} \equiv \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}) \,, \qquad \text{for } a = 1, \dots, k \,. \tag{1.3}$$

Show that

- $\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}$,
- $\{\Gamma^{a+}, \Gamma^{b+}\} = \{\Gamma^{a-}, \Gamma^{b-}\} = 0$,
- $(\Gamma^{a+})^2 = (\Gamma^{a-})^2 = 0$.

(c) These operators thus satisfy raising and lowering anti commutation relations. To construct a representation of the algebra, argue that you can start with a spinor ξ such that $\Gamma^{a-}\xi = 0 \;\forall\, a$ and act with the Γ^{a+} on it. What is the dimensionality of this representation?

(d) All these states assemble into a *Dirac spinor* that can be written like

$$|s\rangle = |s_0, s_1, \dots, s_k\rangle = (\Gamma^{k+})^{s_k+\frac{1}{2}} \dots (\Gamma^{0+})^{s_0+\frac{1}{2}}\xi \,, \qquad \text{with} \quad s_j = \pm\frac{1}{2} \,, \; \forall j \in \{0, \dots, k\} \,. \tag{1.4}$$

Show that

$$S_a \equiv i^{\delta_a}\Sigma^{2a,2a+1} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2} \,, \tag{1.5}$$

and deduce that

$$S_a \,|s\rangle = s_a \,|s\rangle \,. \tag{1.6}$$

(e) Since we have an even number of dimensions, we can define a new gamma matrix

$$\Gamma \equiv i^{-k}\Gamma^0\Gamma^1 \dots \Gamma^{d-1}\,, \tag{1.7}$$

which satisfies

$$\Gamma^2 = 1\,, \qquad \{\Gamma, \Gamma^\mu\} = 0\,, \qquad [\Gamma, \Sigma^{\mu\nu}] = 0\,, \qquad \Gamma = 2^{k+1} S_0 S_1 \dots S_k. \tag{1.8}$$

Show that these properties are satisfied in the specific case $d = 2$ ($k = 0$) and convince yourself that they are true in general.

(f) Deduce how Γ acts on a Dirac spinor. It allows us to decompose generic Dirac spinors into positive or negative chirality *Weyl spinors*.

Comment: In odd dimensions $d = 2k + 3$ we can add Γ to the set of Γ^μ in one dimension less and altogether they satisfy the Clifford algebra in $2k+3$ dimensions. The dimensionality is thus the same as in $2k + 2$ dimensions for a Dirac spinor. In this case though, Γ does not commute with $\Sigma^{\mu,d-1}$ and the Dirac representation is thus irreducible. Chirality is not meaningful in odd dimensions. Also note that the Majorana condition is possible only if $d = 0, 1, 2, 3, 4$ mod 8. Majorana-Weyl spinors are allowed if $d = 2$ mod 8.

2 Worldsheet SUSY I

Consider the superspace Majorana spinor coordinate

$$\theta \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{2.1}$$

which satisfies $\theta^* = \theta$. We have $\bar{\theta} \equiv \theta^\dagger \gamma^0 = \theta^T \gamma^0$ where

$$\gamma^0 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.2}$$

a) Compute explicitly $\bar{\theta}$ and show that the following Fierz identity holds:

$$\theta_\alpha \bar{\theta}^\beta = -\frac{1}{2}\delta_\alpha^\beta \bar{\theta}^\gamma \theta_\gamma\,. \tag{2.3}$$

b) Compute explicitly $\bar{\theta}\theta$ and deduce

$$\frac{\partial}{\partial \bar{\theta}^\alpha}\bar{\theta}\theta = 2\theta_\alpha\,, \qquad \frac{\partial}{\partial \theta_\alpha}\bar{\theta}\theta = -2\bar{\theta}^\alpha \qquad \text{and} \qquad \int \mathrm{d}^2\theta\, \bar{\theta}\theta = -2i. \tag{2.4}$$

c) We have

$$\begin{aligned} Q_\alpha &\equiv \frac{\partial}{\partial \bar{\theta}^\alpha} + i(\gamma^a\theta)_\alpha \partial_a\,, \qquad & \bar{Q}^\alpha &\equiv -\frac{\partial}{\partial \theta_\alpha} - i(\gamma^a\theta)^\alpha \partial_a\,, \\ D_\alpha &\equiv \frac{\partial}{\partial \bar{\theta}^\alpha} - i(\gamma^a\theta)_\alpha \partial_a\,, \qquad & \bar{D}^\alpha &\equiv -\frac{\partial}{\partial \theta_\alpha} + i(\gamma^a\theta)^\alpha \partial_a\,. \end{aligned} \tag{2.5}$$

Check that $\{D_\alpha, Q_\beta\} = 0$.

d) Check that $\int \mathrm{d}^2\theta \frac{\partial}{\partial\theta_\alpha} f(\theta) = 0$ for any function f and recall why it ensures that integrals over the whole superspace of any function of superfields and their supercovariant derivatives provides a SUSY invariant quantity.

3 Worldsheet SUSY II

In light of the previous exercise, the superstring action is proposed to be

$$S = \frac{i}{4\pi} \int \mathrm{d}^2\sigma \mathrm{d}^2\theta (\bar{D}^\alpha Y^\mu)(D_\alpha Y_\mu)\,, \tag{3.1}$$

where the superfield $Y^\mu(\sigma, \theta)$ is defined in components thanks to the Taylor expansion

$$Y^\mu(\sigma, \theta) \equiv X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma)\,. \tag{3.2}$$

a) Use the results of the previous exercise to compute the supercovariant derivatives and perform the integral over θ. You should find

$$S = -\frac{1}{2\pi} \int \mathrm{d}^2\sigma (\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \gamma^a \partial_a \psi_\mu - B^\mu B_\mu)\,. \tag{3.3}$$

b) Compute the variation of the superfield $\delta_\epsilon Y^\mu(\sigma, \theta) = \bar{\epsilon} Q Y^\mu(\sigma, \theta)$ and use the definition

$$\delta_\epsilon Y^\mu(\sigma, \theta) \equiv \delta X^\mu(\sigma) + \bar{\theta}\delta\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta \delta B^\mu(\sigma)\,, \tag{3.4}$$

to identify the off-shell SUSY transformations in components.