String theory lecture - Exercise sheet 11

To be discussed on January 15th

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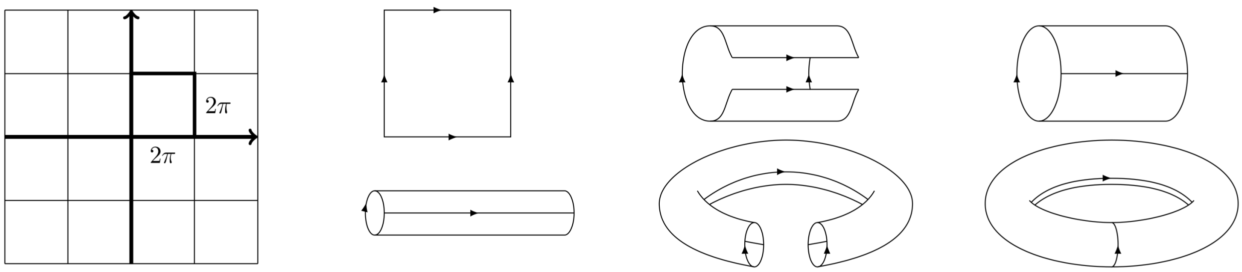
The goal of this exercise sheet is to the closed-string the one-loop vacuum amplitude. We prepare our dive into the computation by exploring the moduli space of the torus. After that, we derive the partition function of the closed string and study its crucial properties by comparing it with a field theory result. For the derivation of the partition function you can refer to the lecture notes by Timo Weigand and David Tong that are available on the lecture website.

1 One-loop warm-up: The torus moduli space

Before we evaluate amplitudes at one-loop for the closed string, we need to know how to integrate over conformally inequivalent metrics on the torus: We need to uncover the *torus moduli space*. The torus has by definition the topology of $S^1 \times S^1$ and is parametrized by

$$(\sigma_1, \sigma_2) \cong (\sigma_1, \sigma_2) + 2\pi(m, n)\,, \qquad m, n \in \mathbb{Z}\,. \tag{1.1}$$

This identification gives rise to a lattice in $\mathbb{R}^2$ that describes the torus (see image below).



a) The *Riemann-Roch theorem* relates the number of metric moduli μ of a Riemann surface to the number of conformal Killing vectors κ. The theorem states

$$\mu - \kappa = -3\chi\,. \tag{1.2}$$

- Argue that the theorem is in agreement with what you know about the sphere S^2.
- The torus being simply the product of two circles, guess what its conformal group is and deduce from the Riemann-Roch theorem that it should have only one complex modulus.

b) We already know (see lecture and exercise sheet 2) that conformal transformations in 2d allow us to have locally flat space. A non-trivial topology generally prevents the map to flat space from being well-defined globally but, because $\chi = 0$ for the torus, in this case it is.

- Defining $z \equiv \sigma_1 + i\sigma_2$ one can thus reach the metric $\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}$. But in the process, the original periodicity may be modified to shifts by arbitrary vectors u_a and v_a, $a \in \{1,2\}$. Think of the lattice of the torus defined above to be generated by these two vectors u_a and v_a instead of $2\pi(1,0)$ and $2\pi(0,1)$. Argue that you can fix u_a without loss of generality so that you are left only with v_a.
- Define $\tau \equiv v_1 + iv_2$: This is the complex modulus we were looking for. The torus has a flat metric but the periodicity involves the modulus. Another way to look at the situation is to redefine the coordinate z to recover the original periodicities. Do this, and then τ appears in the metric

$$\mathrm{d}s^2 = \underbrace{\mathrm{d}z\mathrm{d}\bar{z}}_{\tau \text{ in periodicties}} = \underbrace{|\mathrm{d}\sigma_1 + \tau\mathrm{d}\sigma_2|^2}_{\text{orginial periodicities}} . \tag{1.3}$$

c) We have identified the torus complex modulus but it does not mean that any value for τ corresponds to inequivalent tori. We must get rid of redundancies to uncover the torus *moduli space*.

- Show that the transformations

$$S: \ \tau \to \tau + 1 , \quad T: \ \tau \to -\frac{1}{\tau} , \quad U: \ \tau \to \frac{\tau}{\tau+1} , \tag{1.4}$$

  appropriately combined with conformal transformations on z, leave the torus invariant.
- With these transformations (actually with only S and T) you can generate the action

$$\tau \to \frac{a\tau + b}{c\tau + d} , \quad \text{where} \quad ad - bc = 1 , \quad a, b, c, d \in \mathbb{Z} . \tag{1.5}$$

  We have already encountered similar sets of transformations. What group does it correspond to (remember to mod out by the transformation $(a, b, c, d) \to (-a, -b, -c, -d)$)?

d) Now that we know the invariance group of the torus, we can describe the sets of values that really describe inequivalent tori and they correspond to the moduli space of the torus. Use the S and T transformations to show that any τ in the upper-half plane can be brought to the *fundamental domain* $\mathcal{F}$ (the lower-half plane is obviously redundant):

$$\mathcal{F} \equiv \left\{ \tau \in \mathbb{C} , \ -\frac{1}{2} \le \mathrm{Re}\,\tau \le \frac{1}{2} , \ |\tau| \ge 1 \right\} . \tag{1.6}$$

Represent $\mathcal{F}$ graphically with proper identifications of its boundaries.

2 The partition function for closed strings

a) In this exercise we want to compute the simplest one-loop amplitude Z_{T^2}, i.e. without any operator insertion. The amplitude involves an integration over the torus moduli space and we first need to set a proper integration measure. One needs the notion of distance between metrics on a manifold M which in general takes the form

$$\mathrm{d}s^2_{\text{metric}} \equiv \int_M \sqrt{\det h}\,\delta h_{ab}\delta h_{cd} h^{bc} h^{da} \,. \tag{2.1}$$

Specialized to the torus case we have

$$\mathrm{d}s^2_{\text{metric}} = \int_{T^2} \sqrt{\det h}\,\mathrm{Tr}\left(\partial_\tau h h^{-1} \partial_{\bar\tau} h h^{-1}\right) \mathrm{d}\tau \mathrm{d}\bar\tau \,. \tag{2.2}$$

With the metric on the torus uncovered in the previous exercise, compute this to show that (ignoring prefactors)

$$\mathrm{d}s^2_{\text{metric}} \sim \frac{\mathrm{d}\tau\mathrm{d}\bar\tau}{\tau_2} \,. \tag{2.3}$$

b) The one-loop vacuum amplitude thus takes the following form:

$$Z_{T^2} \sim \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} \langle 1 \rangle_{\tau, \otimes_{i=1}^D X^i} \cdot \langle \text{ghost insertions} \rangle_{\tau, bc} \equiv \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} Z(\tau) \cdot \langle \text{ghost insertions} \rangle_{\tau, bc} \,. \tag{2.4}$$

We will focus on the X-CFT piece $Z(\tau) \equiv \langle 1 \rangle_{\tau, \otimes_{i=1}^D X^i}$ and simply give the result later for the contribution of the bc-CFT.

Note: It is actually very subtle how to compute properly the vacuum amplitude. Indeed, when there are not enough insertion points, the generic amplitude formula that you have seen in the lecture cannot be applied directly. In addition to carefully writing the moduli space metric when integrating, one should also properly write the conformal Killing group volume and properly compute the "ghost volume". Altogether, these effects produce a total τ_2-dependent factor which turns out to be the $1/\tau_2$ that we already have now in (2.4). We thus trust that no further *subtle* τ_2 factors will arise.

Assume first a rectangular torus, i.e. $\tau = i\tau_2$, and interpret the path integral $\int \mathcal{D}X e^{-S_X}$ as follows: Start from the vacuum, create a state, evolve it by Euclidean time $2\pi\tau_2$ with the Hamiltonian H, identify in- and out-state, sum over states. Translate this into a formula and interpret it as a partition function.

c) Now consider the generic case $\tau = \tau_1 + i\tau_2$. As seen in the previous exercise, this corresponds to a tilt in the torus lattice. Thus, interpret the presence of τ_1 as a spatial translation implemented by a translation operator P before looping back and modify the formula for $Z(\tau)$ accordingly.

d) The Hamiltonian on the cylinder and the translation operator are given by

$$H = \frac{2\pi}{l}(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}) \,, \qquad P = \frac{2\pi}{l}(L_0 - \bar{L}_0) \,. \tag{2.5}$$

Fix $l = 2\pi$ and focus on only one direction with $c = \bar{c} = 1$ to find

$$Z(\tau)^{(1)} = \mathrm{Tr}\, q^{L_0^{(1)} - 1/24} \bar{q}^{\bar{L}_0^{(1)} - 1/24} \,, \qquad \text{with} \qquad q \equiv e^{2i\pi\tau} \,. \tag{2.6}$$

The superscript (1) simply denotes that we only consider one dimension.

e) Using

$$L_0^{(1)} = \frac{\alpha'}{4}p^2 + N^{(1)}\,, \qquad \bar{L}_0^{(1)} = \frac{\alpha'}{4}p^2 + \bar{N}^{(1)}\,, \tag{2.7}$$

Split the trace into a momentum integral and a trace denoted Tr′ which amounts to sum only over oscillators to obtain

$$Z(\tau)^{(1)} = V_1 (q\bar{q})^{-\frac{1}{24}} \int \frac{\mathrm{d}p}{2\pi} e^{-\pi\tau_2\alpha' p^2} \mathrm{Tr}'\, q^{N^{(1)}} \bar{q}^{\bar{N}^{(1)}}\,, \tag{2.8}$$

where $V_1 \equiv \delta(p-p)$ is the spacetime volume for one direction.

f) Evaluate the trace and compute the Gaussian integral to get

$$Z(\tau)^{(1)} = V_1 \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \frac{1}{(q\bar{q})^{1/24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n}\,. \tag{2.9}$$

g) Combine the contributions from the 26 dimensions and use the fact that the ghost part yields a factor $|\eta(\tau)|^4$ to obtain

$$Z_{\tau_2} \sim iV_{26} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} (4\pi^2\alpha'\tau_2)^{-13} |\eta(\tau)|^{-48}\,, \tag{2.10}$$

where the Dedekind η function is defined as

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)\,. \tag{2.11}$$

What is the net effect of the ghosts?

Note for later that if we keep the trace and take the ghosts into account we can write

$$Z_{T^2} = iV_D \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{D}{2}} \frac{1}{q\bar{q}} \mathrm{Tr}'\, q^{N_\perp} \bar{q}^{\bar{N}_\perp}\,, \tag{2.12}$$

where $N_\perp$ counts the transverse oscillators.

h) Check the *modular invariance* of the partition function, i.e. invariance under $PSL(2,\mathbb{Z})$. To do this use the following modular properties of the Dedekind function:

$$\eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau)\,, \qquad \eta(\tau+1) = e^{i\frac{\pi}{12}}\eta(\tau)\,. \tag{2.13}$$

i) To emphasize some key properties of the partition function, let us compare it with its field theory counterpart. Consider a free massive scalar field ϕ with mass m in D dimensions. Recall that the ϕ path integral gives

$$Z \equiv \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int \mathrm{d}^D x \phi(-\partial^2 + m^2)\phi\right) = \exp\left(-\frac{V_D}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D} \ln\left[\frac{1}{2}(p^2+m^2)\right]\right)\,. \tag{2.14}$$

j) From this we extract the vacuum amplitude Z_{S^1} for a single particle, analogue to our torus partition function:

$$Z_{S^1} = -\frac{V_D}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D} \ln\left[\frac{1}{2}(p^2+m^2)\right] . \tag{2.15}$$

Using the Schwinger parametrization

$$\frac{1}{x} = \int \mathrm{d}l e^{-lx} \quad \Longrightarrow \quad -\ln(x) = \int_0^{+\infty} \mathrm{d}l \frac{e^{-xl}}{l} , \tag{2.16}$$

we get

$$Z_{S^1} = V_D \int \frac{\mathrm{d}^D p}{(2\pi)^D} \int_0^{+\infty} \frac{\mathrm{d}l}{2l} e^{-\frac{1}{2}(p^2+m^2)l} . \tag{2.17}$$

But doing this manipulation, the UV divergence at high p has been mapped to a divergence at small l. Interpret the different elements in this formula in a way analogous to our string computation. In particular, what are the roles of $\frac{1}{2}(p^2+m^2)$, l in the exponential and $\int_0^{+\infty} \frac{\mathrm{d}l}{2l}$?

k) Perform the p integral and take into account an infinite spectrum like we have in string theory by introducing a trace to get

$$Z_{S^1} = iV_D \int_0^{+\infty} \frac{\mathrm{d}l}{2l^{1+\frac{D}{2}}} (2\pi)^{-\frac{D}{2}} \operatorname{Tr}' e^{-m^2 l/2} . \tag{2.18}$$

l) For the strings we have

$$m^2 = \frac{2}{\alpha'}(N_\perp + \bar{N}_\perp - 2) , \qquad N_\perp = \bar{N}_\perp . \tag{2.19}$$

To match the partition function with the string expression, implement the level-matching by introducing

$$\delta_{L_0,\bar{L}_0} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}s e^{2i\pi s(N_\perp - \bar{N}_\perp)} . \tag{2.20}$$

and define $\tau \equiv s + \frac{2li}{\alpha'}$. You should get

$$Z_{S^1} = iV_D \int_{\text{strip}} \frac{\mathrm{d}^2\tau}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{D}{2}} \frac{1}{q\bar{q}} \operatorname{Tr}' q^{N_\perp} \bar{q}^{\bar{N}_\perp} . \tag{2.21}$$

This is exactly what we had for the string, with the crucial difference that the integration is over the strip $\{\tau \in \mathbb{C}\, ,\ \tau_2 \geq 0\, ,\ |\tau_1| \leq \frac{1}{2}\}$.

m) Draw the strip and the fundamental domain on a same graphic. What happens to the partition function when $\tau_2 \to 0$? Conclude about the uttermost importance of modular invariance in string theory.