String theory lecture - Exercise sheet 1

General relativity as a quantum field theory

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The goal of this exercise sheet is to recall important facts about General Relativity (GR) on the one hand and gauge theories on the other hand. Armed with these reminders, we can describe gravity from a Quantum Field Theory (QFT) perspective and explore the crucial differences of this theory compared to the Yang-Mills theories you are used to. These differences force us to go beyond QFT to describe the gravitational interaction and motivate the elaboration of string theory. The last exercise explores properties of the "Nambu-Goto" and "Polyakov" actions for relativistic particles.

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1 Yang-Mills theories in a nutshell

1.1 Abelian case

For a $U(1)$ gauge field coupled to a massive fermionic Dirac field ψ, we have the following Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \,, \tag{1.1}$$

where g is the coupling constant and $\not{D} \equiv \gamma^\mu D_\mu$.

a) Recall the definition of the covariant derivative D_μ and that of the field-strength tensor $F_{\mu\nu}$ as a commutator involving the covariant derivative. Give an expanded expression for the field strength in terms of the gauge field.

1.2 Non-Abelian theories (QCD-like)

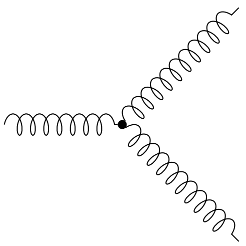
Analogously to the previous case, for an $SU(N)$ gauge theory we have the Lagrangian

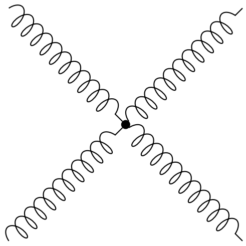
$$\mathcal{L} = -\frac{1}{4g^2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \,, \tag{1.2}$$

where now ψ is a vector of $SU(N)$.

a) Recall how to generalize the Abelian case and observe that, even without ψ, there are interactions.

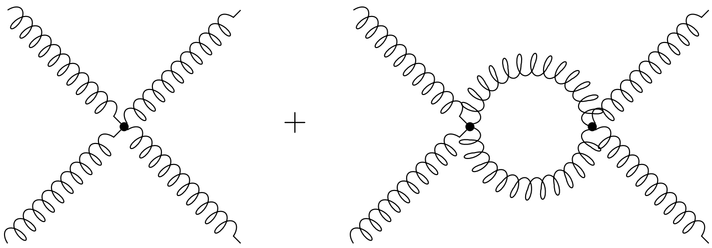
b) Derive symbolically (i.e. without numerical factors and detailed index structure) the Feynman rules for the two following vertices:





Don't forget to apply a field redefinition of the form $A_\mu \to g A_\mu$.

c) Use the Feynman rules to give a symbolic expression for the next-to-leading order (NLO) term in the following expansion:



$$= \quad \text{LO} + \text{NLO}$$

Evaluate how the momentum integral diverges.

d) What is the mass dimension of the Yang-Mills coupling in d dimensions? Comment on the $d = 4$ case.

2 A QFT for gravity

We now want to understand GR as a QFT by mirroring the Yang-Mills construction. Let us proceed step by step.

2.1 Gravity as a gauge theory

a) What group do you think we want to gauge? What is the gauge field? What spin has it? How do we call it?

b) In GR the covariant derivative is defined as follows (acting on a vector V)

$$(D_\mu V)^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu} V^\rho \,, \tag{2.1}$$

where the Christoffel symbols are expressed in terms of the metric $g_{\mu\nu}$ like

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{g^{\rho\sigma}}{2} \left(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma} \right) \,. \tag{2.2}$$

We have used the notation $g_{\mu\nu,\sigma} \equiv \partial_\sigma g_{\mu\nu}$.

In analogy with the Yang-Mills theory, define a field-strength tensor $R_{\mu\nu\rho}{}^{\sigma}$ such that $R_{\mu\nu\rho}{}^{\sigma} V^\rho \equiv ([D_\mu, D_\nu] V)^\sigma$ and express it in terms of the Christoffel symbols.

c) How do we call this tensor and what does it measure? Also recall its symmetries. What is different compared to Yang-Mills theories?

d) We define the Ricci tensor as $R_{\mu\nu} \equiv R_{\rho\mu\nu}{}^{\rho}$ and the Ricci scalar as $\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu}$. From this the Einstein-Hilbert action for GR is given by

$$S_{\text{EH}} = \frac{M_{\text{p}}^2}{2} \int \sqrt{-g} \mathcal{R} \,. \tag{2.3}$$

The Einstein-Hilbert Lagrangian is nothing else but the kinetic term of the gauge field, built from the field strength. In this case what is the gauge coupling and what dimension has it?

2.2 Perturbation (weak-field) theory

The next step is to expand the gauge field as a perturbation around a background and explore the properties of the QFT that one obtains upon quantization of the fluctuations. We thus write the metric

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{2.4}$$

where $\eta_{\mu\nu}$ is the Minkowski background and $h_{\mu\nu}$ encodes small perturbations. Let us try to understand the structure of the resulting QFT, its Feynman rules and its properties. We will focus on "pure gravity" to emphasize key differences with Yang-Mills theories but of course one could also be interested in coupling this QFT to matter fields.

a) As we said before, the Einstein-Hilbert (EH) action is the kinetic term of the gauge field. We want to expand this term in powers of $h_{\mu\nu}$ to read the Feynman rules. To do this, follow these steps:

- Express the Ricci scalar in the EH action in terms of the Christoffel symbols.
- Integrate by part the two derivative terms to find (discard the total derivatives)

$$S_{\text{EH}} = \int \mathrm{d}^4x \sqrt{-g} g^{\mu\nu} \left(\Gamma_{\mu\rho}{}^{\sigma} \Gamma_{\nu\sigma}{}^{\rho} - \Gamma_{\rho\sigma}{}^{\sigma} \Gamma_{\mu\nu}{}^{\rho} \right) . \tag{2.5}$$

  You will need to use the following relations:

$$\partial_\rho \sqrt{-g} = \sqrt{-g} \Gamma_{\rho\sigma}{}^{\sigma} \,, \qquad \partial_\rho g^{\mu\nu} = -\Gamma_{\rho\sigma}{}^{\mu} g^{\sigma\nu} - \Gamma_{\rho\sigma}{}^{\nu} g^{\mu\sigma} \,. \tag{2.6}$$

- If you inserted the definition of the Christoffel symbols in terms of the metric (don't feel obliged to do it) you would find

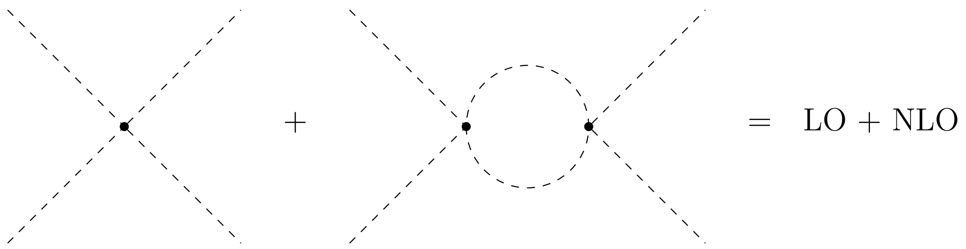
$$S_{\text{EH}} = -\int \mathrm{d}^4x \frac{\sqrt{-g}}{4} \left(2g^{\sigma\gamma} g^{\rho\delta} g^{\alpha\beta} - g^{\gamma\delta} g^{\alpha\beta} g^{\rho\sigma} - 2g^{\sigma\alpha} g^{\gamma\rho} g^{\delta\beta} + g^{\rho\sigma} g^{\alpha\gamma} g^{\beta\delta} \right) g_{\alpha\beta,\rho} g_{\gamma\delta,\sigma} \tag{2.7}$$

- Now you can easily and explicitly express the $\mathcal{O}(h^2)$ term.

b) What symbolic structure do you expect for higher-order terms and why? Do you see how it is different from the Yang-Mills terms?

c) Draw the two first vertices and write the symbolic momentum-dependence that you expect (don't forget to rescale the gauge field in an analogous manner as in the Yang-Mills case).

d) Mirroring the Yang-Mills analysis of the previous exercise, use the Feynman rules to give a symbolic expression for the next-to-leading order (NLO) term in the following expansion (dashed lines represent our gauge field at hands):



Evaluate how the momentum integral diverges.

e) What does this imply about quantum gravity?

f) Conclude that the string theory lecture is very well motivated and that you are very excited to plunge into the entrails of the theory.

3 Relativistic particles

a) Show that the Nambu-Goto action

$$S_{\rm NG} = -m \int \mathrm{d}\tau \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}\,, \quad \mu,\nu = 0,\ldots,d-1\,, \tag{3.1}$$

of the relativistic particle is invariant under arbitrary reparametrization $\tau \to \tau'(\tau)$.

b) Derive the non-relativistic limit

$$S_{\rm NG} \simeq \int \mathrm{d}t \left(\frac{m}{2}\vec{v}^2 - m\right)\,. \tag{3.2}$$

c) Demonstrate the reparametrization invariance ($\tau \to \tau'(\tau)$) of the Polyakov action

$$S_{\rm p} = -\frac{m}{2}\int \mathrm{d}\tau \sqrt{-h_{\tau\tau}}\left(h_{\tau\tau}^{-1}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X_\mu}{\mathrm{d}\tau} + 1\right)\,. \tag{3.3}$$

d) Derive the Nambu-Goto action (3.2) from the Polyakov form (3.3) as sketched in the lecture, i.e. use the equations of motion for $h_{\tau\tau}$ to eliminate $h_{\tau\tau}$ in (3.3).